

ON DILATIONS AND TRANSFERENCE
FOR CONTINUOUS ONE-PARAMETER
SEMIGROUPS OF POSITIVE
CONTRACTIONS ON \mathcal{L}^p -SPACES.

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Acknowledgements : These are notes of some lectures held at the Institute of Mathematics of the University of Wrocław in November and December 1996. There the lectures were presented under the title “Functional calculus of operators on Hilbert spaces and on other classes of Banach spaces”.

The author is grateful to Professor M. Bożejko for having had the possibility to present the material at his seminar. He thanks the audience for their interest.

We thank the referee for his comments. They were quite useful for improving the English as well as the mathematical presentation of the contents of this note.

Last not least we take this chance to thank Ms Georg from the mathematical institute of the University of the Saarland for putting to L^AT_EX parts of the manuscript.

Pre-Introduction

During the course held at the Wrocław University things developed a bit different from what was originally planned. Following the appearing interests we partially changed the subject and now the title too, according to what has been presented.

We still found it worthwhile to include some remarks on functional calculus on Hilbert spaces which now are presented in the introduction. Readers interested only in one-parameter semigroups on \mathcal{L}^p -spaces may thus leave out the introduction. Those who are only interested in Hilbert spaces may read the introduction and skip the rest.

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Chapter 1

Introduction

1.1 Norm Estimates for Polynomial Calculus

Let H be a Hilbert space, $T : H \longrightarrow H$ a linear contraction, i.e. T is a linear map of norm at most one. In 1951 J. von Neumann [40] proved that for any polynomial

$$p(z) = \sum_{n=0}^N a_n z^n, \quad z \in \mathbb{C}$$

the inequality

$$\|p(T)\| \leq \sup_{|z| \leq 1} |p(z)|$$

holds true. Here, of course, we denote $p(T) = \sum_{n=0}^N a_n T^n$ and consider it as an element of the bounded linear operators $B(H)$ of H .

Let me mention a short proof of this. First, if $U \in B(H)$ is a unitary operator, thus a very special contraction, then

$$\begin{aligned} \|p(U)\|^2 &= \|p(U)^* p(U)\| \\ &= r(p(U)^* p(U)) \\ &= \sup\{|\lambda| : \lambda \in \sigma(p(U)^* p(U))\}, \end{aligned}$$

where $\sigma(A) = \{\lambda : \lambda - A \notin \text{Inv}(B(H))\}$ denotes the spectrum of the linear operator A and $r(A)$ its spectral radius.

On the other hand,

$$\begin{aligned} p(U)^* p(U) &= \sum_{n \in \mathbb{N}} \bar{a}_n (U^*)^n \cdot \sum_{n \in \mathbb{N}} a_n U^n \\ &= \sum_{l=-N}^N \sum_{\{n, l : n+k=l\}} \bar{a}_n a_k U^l \\ &= F(U), \end{aligned}$$

where $F(z) = \sum_{l=-N}^N \sum_{\{n,l:n+k=l\}} \bar{a}_n a_k z^l$, $z \in \mathbb{C} \setminus \{0\}$ is a rational function which is holomorphic in a neighbourhood of $\sigma(U) \subset \{z \in \mathbb{C} : |z| = 1\}$.

By the spectral mapping theorem $\sigma(F(U)) \subset F(\sigma(U))$, and thus

$$\begin{aligned} r(p(U)^* p(U)) &= \sup\{|\lambda| : \lambda \in \sigma(F(U))\} \\ &= \sup\{|F(\lambda)| : |\lambda| = 1\} \\ &= \sup\{|\overline{p(z)} p(z)| : |z| = 1\} \\ &= \sup\{|p(z)| : |z| = 1\}^2, \end{aligned}$$

which establishes von Neumann's inequality in this case. □

Now, Foias and Sz. Nagy [51] constructed a unitary dilation for an arbitrary contraction:

Theorem 1.1.1 (Sz. Nagy-Foias) *Given a contraction $T \in B(H)$ there exists a Hilbert space K together with a unitary $U \in B(K)$, an isometric embedding $I : H \rightarrow K$ and an orthogonal projection $P : K \rightarrow H$ such that for $k = 0, 1, \dots$*

$$T^k = P \circ U^k \circ I.$$

Using this theorem we obtain, for any contraction T ,

$$\begin{aligned} p(T) &= \sum_{n=0}^N a_n T^n \\ &= P \circ p(U) \circ I, \end{aligned}$$

where P , U , and I are as in the theorem. Thus,

$$\begin{aligned} \|p(T)\| &\leq \|p(U)\| \\ &\leq \sup\{|p(z)| : |z| = 1\}, \end{aligned}$$

which is von Neumann's inequality in the general case. □

It is clear, that $T' \in B(H)$ is polynomially bounded if it is similar to a contraction T by a bounded operator $S \in \text{Inv}(B(H))$, i.e. if $T' = S^{-1}TS$. Since in this case for any polynomial p : $p(T') = S^{-1}p(T)S$, and hence

$$\begin{aligned} \|p(T')\| &\leq \|S^{-1}p(T)S\| \\ &\leq \|S^{-1}\| \|S\| \|p(T)\|_\infty. \end{aligned}$$

A natural question is whether the converse holds true too. This is known as the Halmos problem [25]. But it has some history: Sz. Nagy [49] [50] in 1947 and 1959 respectively proved

- if $T \in B(H)$ is invertible and T and its inverse are power-bounded, that is, if

$$\sup\{\|T^n\| : n \in \mathbb{Z}\} < \infty,$$

then T is similar to a unitary operator.

- if $T \in B(H)$ is compact and power-bounded, for positive potencies only, i.e. if

$$\sup\{\|T^n\| : n \in \mathbb{N}\} < \infty,$$

then T is similar to a contraction.

His question then was:

If a general operator $T \in B(H)$ is power-bounded, is then T similar to a contraction?

- Foguel [21] gave a counterexample in 1964.
- Lebow [33] proved in 1968 that the operator, constructed by Foguel, is not even polynomially bounded.
- Bożejko [9] 1987 produces a whole class of examples of power-bounded not polynomially bounded operators.
- In between, Peller [42] 1982 studied the space of functions which act on power-bounded Hilbert space operators.
- Halmos' question finitely has been answered in the negative by Pisier [44] 1996, only recently.
- An extension of Pisier's example has been obtained by Davidson and Paulsen [17] 1997.

1.2 Complete Boundedness

For $n \in \mathbb{N}$ let $M_n = M_n(\mathbb{C}) = B(l_n^2)$ denote the space of complex $n \times n$ matrices normed as acting on the complex n dimensional Hilbert space l_n^2 .

For $T \in B(H)$ a requirement which is stronger than its polynomial boundedness is that of its complete polynomial boundedness.

Definition 1.2.1 *An operator $T \in B(H)$ is called completely polynomially bounded, if there exists $C > 0$, such that for all $n \in \mathbb{N}$ and all $n \times n$ matrices of polynomials $(p_{i,j})_{i,j=1}^n$*

$$\left\| (p_{i,j}(T))_{i,j=1}^n \right\|_{M_n \otimes B(H)} \leq C \sup \left\{ \left\| (p_{i,j}(z))_{i,j=1}^n \right\|_{M_n} : |z| \leq 1 \right\}.$$

Here a matrix $a = (a_{i,j})_{i,j=1}^n$ of operators $a_{ij} \in B(H)$ acts on

$$l_n^2(H) = l_n^2 \oplus_2 H = \left\{ v = (v_1, \dots, v_n) : v_i \in H, \|v\| = \left(\sum_{i=1}^n \|v_i\|^2 \right)^{\frac{1}{2}} \right\}$$

by

$$a(v)_i = \sum_{j=1}^n a_{ij} v_j.$$

Based on the Sz. Nagy-Foias dilation theorem it is not hard to show that any contraction $T \in B(H)$, and hence any operator $T' \in B(H)$ which is similar to a contraction, is completely polynomially bounded. Building on work of Wittstock [54], Arveson [4], [5] and Haagerup [24] Paulsen [41] showed the converse:

Theorem 1.2.1 (Paulsen) *An operator $T' \in B(H)$ is similar to a contraction if and only if T' is completely polynomially bounded.*

One more Definition:

Definition 1.2.2 *If K is another Hilbert space, $A \subset B(K)$ a linear subspace and $\varphi : A \rightarrow B(H)$ a linear map, then φ is called completely bounded, if there exists $C > 0$ such that for all $n \in \mathbb{N}$ and all $n \times n$ matrices $a = (a_{i,j})_{i,j=1}^n \in M_n(A)$*

$$\left\| (\varphi(a_{i,j}))_{i,j=1}^n \right\|_{B(l_n^2 \otimes_2 H)} \leq C \left\| (a_{i,j})_{i,j=1}^n \right\|_{B(l_n^2 \otimes_2 K)}. \quad (1.1)$$

We denote $\|\cdot\| : \varphi \mapsto \|\varphi\|$ the associated norm, i.e.

$$\|\varphi\| = \inf \{ C : (1.1) \text{ holds true for all } a \in M_n(A) \text{ and all } n \in \mathbb{N} \}.$$

In our case, with $S = \{z \in \mathbb{C} : |z| = 1\}$ and σ the surface measure on S , let $K = L^2(S, \sigma)$ and consider the algebra of all polynomials

$$A = \{p : p - \text{polynomial}\} \subset B(K)$$

as embedded by means of its action of pointwise multiplication on $L^2(S, \sigma)$ -functions:

$$\begin{aligned} f &\mapsto p \cdot f, \\ \text{where for all } p &\in A, \quad f \in L^2(S, \sigma) \\ p \cdot f(z) &= p(z)f(z), \quad z \in S \end{aligned}$$

denotes the pointwise product of functions.

Then $T \in B(H)$ is completely polynomially bounded, exactly if the corresponding homomorphism

$$p \mapsto p(T), \quad p \in A$$

from A into $B(H)$ is completely bounded.

1.3 Analytic Semigroups on Hilbert Space

It is obvious that the above mentioned problems have analogues for one-parameter semigroups instead of the discrete semigroup \mathbb{Z}_+ only.

A C_0 -semigroup $(T_t)_{t \geq 0}$ acting on H has an infinitesimal generator

$$Ax = \lim_{t \searrow 0} \frac{T_t - 1}{t} x, \quad x \in D(A) := \{ \tilde{x} : \text{the limit } \lim_{t \searrow 0} \frac{T_t - 1}{t} \tilde{x} \text{ exists} \}$$

which is a closed densely defined operator.

We shall consider here only semigroups which admit bounded analytic extensions to some nontrivial cone $\Gamma_\theta = \{z \neq 0 : |\arg(z)| < \theta\} \subset \mathbb{C}$. (Analytic refers to weak analyticity, i.e. a map $\Phi : \Gamma \rightarrow B(H)$ is called analytic, may be sometimes holomorphic, if for all $x, y \in H$ $z \mapsto (\Phi(z)x, y)$ is analytic on Γ .)

Definition 1.3.1 *A C_0 -semigroup $(T_t)_{t \geq 0}$ acting on H admits a bounded analytic extension to some cone Γ_θ if there exists a map*

$$\begin{aligned} T : \Gamma_\theta &\rightarrow B(H) \\ z &\mapsto T_z, \end{aligned}$$

analytic on Γ_θ and extending $T : t \mapsto T_t$, such that

$$\begin{aligned} T_{z+z'} &= T_z T_{z'} \quad \forall z, z' \in \Gamma_\theta, \\ \lim_{z \rightarrow 0, z \in \Gamma_\theta} T_z x &= x \quad \forall x \in H \end{aligned}$$

and

$$\sup_{z \in \Gamma_\theta} \|T_z\| < \infty.$$

Assume that A is injective then, for $z \neq 0, 0 \leq \operatorname{Re} z < 1$, complex powers $(-A)^z$ of the negative of A can be defined as closable densely defined operators, fulfilling $(-A)^z \cdot (-A)^{z'} = (-A)^{z+z'}$.

Definition 1.3.2 *We say that $(-A)$ admits bounded imaginary powers if there exist $c > 0, C > 0$ such that $(-A)^{is} \in B(H)$ and*

$$\| (-A)^{is} \| \leq C e^{c|s|} \quad \forall s \in \mathbb{R}.$$

Then in fact, $s \mapsto (-A)^{is}$ is a C_0 -group of operators on H .

Le Merdy [32] connected the similarity problem for analytic semigroups to the functional calculus of the negative of the generator:

Theorem 1.3.1 (Le Merdy) *Let $(T_t)_{t \geq 0}$ be a C_0 -semigroup which admits an analytic extension. Let A be its generator and assume that A is injective. Then $(-A)$ admits bounded imaginary powers if and only if the semigroup is similar to a contraction semigroup.*

The implication “ \Leftarrow ” is entirely due to Prüss and Sohr [45]. For the other implication Le Merdy adapts arguments of McIntosh [37] and Yagi [55] to establish the complete boundedness of the functional calculus for rational functions of degree less than -1 , with poles in the resolvent set of $-A$ and uses a theorem of Paulsen analogous to the one cited above.

Chapter 2

Dilation Theorems

2.1 \mathcal{L}^p -Spaces as Banach Lattices

Since I shall treat an \mathcal{L}^p -space $E = L^p(\Omega, \mu)$ as a Banach lattice some remarks on scalars are in order. Decomposing elements of E , that is, complex valued functions, in their real and imaginary parts we obtain a decomposition, as real linear spaces:

$$L^p(\Omega, \mu; \mathbb{C}) = L^p(\Omega, \mu; \mathbb{R}) \oplus iL^p(\Omega, \mu; \mathbb{R}).$$

Here, on $L^p(\Omega, \mu; \mathbb{R})$, in addition to the linear operations the pointwise operations of maximum and minimum of two elements and the absolute value are defined as

$$\begin{aligned} f \vee g(\omega) &= \max\{f(\omega), g(\omega)\}, \quad \omega \in \Omega, \\ f \wedge g(\omega) &= \min\{f(\omega), g(\omega)\}, \quad \omega \in \Omega, \quad f, g \in L^p(\Omega, \mu; \mathbb{R}), \\ |f|(\omega) &= f \vee (-f)(\omega), \quad \omega \in \Omega, \quad f \in L^p(\Omega, \mu; \mathbb{R}). \end{aligned}$$

Moreover, the norm on $L^p(\Omega, \mu; \mathbb{C})$ is related to the norm on $L^p(\Omega, \mu; \mathbb{R})$ by

$$\|f\| = \left(\int_{\Omega} |f(\omega)|^p d\mu(\omega) \right)^{\frac{1}{p}} = \left(\int_{\Omega} (|\operatorname{Re} f|^2(\omega) + |\operatorname{Im} f|^2(\omega))^{\frac{p}{2}} d\mu(\omega) \right)^{\frac{1}{p}},$$

where $\operatorname{Re} f(\omega) = \frac{1}{2} (f(\omega) + \overline{f(\omega)})$, $\operatorname{Im} f(\omega) = \frac{1}{2} (f(\omega) - \overline{f(\omega)})$ are elements of $L^p(\Omega, \mu; \mathbb{R})$.

One of the first facts to note is:

Proposition 2.1.1 *If $T : L^p(\Omega, \mu; \mathbb{R}) \rightarrow L^p(\Omega, \mu; \mathbb{R})$ is a (real) linear bounded operation, then its canonical extension*

$$T_{\mathbb{C}}(f + ig) = Tf + iTg \quad f, g \in L^p(\Omega, \mu; \mathbb{R})$$

has the same norm bound as T : $\|T_{\mathbb{C}}\| = \|T\|$.

Proof: I don't know, but for me this assertion does not appear completely trivial and thus requires a proof.

For $\alpha, \beta \in \mathbb{R}$ and two independent Gaussian distributed random variables X, Y , with mean zero and variance one, the variables $Z = \alpha X + \beta Y$ and $\tilde{Z} = (\alpha^2 + \beta^2)^{\frac{1}{2}} \cdot X$ are equidistributed. To prove this one computes the Fourier transforms

$$\begin{aligned} \mathcal{E}(\exp it(Z)) &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-it(\alpha x + \beta y)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}|x|^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}|y|^2} dx dy \\ &= \int_{\mathbb{R}} e^{-it\alpha x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}|x|^2} dx \int_{\mathbb{R}} e^{-it\beta y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}|y|^2} dy \\ &= e^{-\frac{1}{2}(t\alpha)^2} e^{-\frac{1}{2}(t\beta)^2} = e^{-\frac{1}{2}t^2(\alpha^2 + \beta^2)} \\ &= \mathcal{E}(\exp(it(\alpha^2 + \beta^2)^{\frac{1}{2}} X)). \end{aligned}$$

Thus for $f, g \in L^p(\Omega, \mu; \mathbb{R})$:

$$\mathcal{E}(|f(\omega)X + g(\omega)Y|^p) = \mathcal{E}\left(|(f^2(\omega) + g^2(\omega))^{\frac{1}{2}} \cdot X|^p\right),$$

and with $\lambda = \mathcal{E}|X|^p$:

$$\int_{\Omega} (|f|^2(\omega) + |g|^2(\omega))^{\frac{p}{2}} d\mu(\omega) = \frac{1}{\lambda} \int_{\Omega} \mathcal{E}|f(\omega)X + g(\omega)Y|^p d\mu(\omega).$$

Let $h = f + ig$, then

$$\begin{aligned} \|T_{\mathbb{C}}h\|^p &= \frac{1}{\lambda} \int_{\Omega} \mathcal{E}|Tf(\omega)X + Tg(\omega)Y|^p d\mu(\omega) \\ &= \frac{1}{\lambda} \mathcal{E} \int_{\Omega} |T(f(\omega)X + g(\omega)Y)|^p d\mu(\omega) \\ &\leq \|T\|^p \frac{1}{\lambda} \mathcal{E} \int_{\Omega} |f(\omega)X + g(\omega)Y|^p d\mu(\omega) \\ &= \|T\|^p \|h\|^p. \end{aligned}$$

From this the inequality $\|T_{\mathbb{C}}\| \leq \|T\|$ is obvious. □

Remark 2.1.1 We used here that a two-dimensional Hilbert space is isometrically isomorphic to a closed subspace of an \mathcal{L}^p -space. This gave the equality of the norms of T and its extension $T_{\mathbb{C}}$. Our proof is a special case of the argument which Marcinkiewicz and Zygmund gave for their famous extension theorem [36]. There, among others, the situation is considered of extending an operator $T : L^p(\Omega, \mu) \rightarrow L^p(\Omega, \mu)$ to an operator T_H defined on the Hilbert space valued function space $L^p(\Omega, \mu; H)$ by extending linearly the definition given on simple tensors

$$T_H : f \otimes \xi \mapsto Tf \otimes \xi,$$

where for $f \in L^p(\Omega, \mu)$, $\xi \in H$ the simple tensor is the H -valued function $f \otimes \xi(\omega) = f(\omega)\xi$, $\omega \in \Omega$.

For a general Banach lattice E and a Hilbert space H there is sense in an extension of the lattice and of an operator $T \in B(E)$. To prove the boundedness of the extended operator one has to invoke Grothendieck's theorem to the result that

$$\|T_H\| \leq K_G \|T\|,$$

where K_G denotes the Grothendieck constant. The reader may find a discussion of these facts in the book [34] of Lindenstrauss and Tzafriri.

Definition 2.1.1 *Now a linear¹ operator*

$$T : L^p(\Omega, \mu; \mathbb{C}) \rightarrow L^p(\Omega, \mu; \mathbb{C})$$

is called positive, $T \geq 0$, if for all $f \in L^p(\Omega, \mu; \mathbb{C})$

$$f \geq 0 \text{ implies } Tf \geq 0.$$

Remark 2.1.2 Clearly, a positive linear operator T on a complex \mathcal{L}^p -space leaves the real subspace $L^p(\Omega, \mu; \mathbb{R})$ invariant, and it is the complexification of its restriction $T_{\mathbb{R}}$.

Example 2.1.1 If $E = L^p(\Omega, \mu)$ is finite dimensional, then, for some n and some $\omega_1, \dots, \omega_n > 0$,

$$E = l_n^p(\omega) = \left\{ \alpha = (\alpha_1, \dots, \alpha_n) \quad : \quad \alpha_i \in \mathbb{C} \quad \left(\sum_{i=1}^n |\alpha_i|^p \omega_i \right)^{\frac{1}{p}} = \|\alpha\| \right\}.$$

For $1 \leq i \leq n$ let $\delta_i = (0, 0, 1, 0, \dots, 0)$, where the nonzero entry is at the i -th position, and let $\{\delta_1, \dots, \delta_n\}$ be the standard basis in \mathbb{C}^n which we usually take as a basis of l_n^p .

When a linear operator T on E is represented by its matrix with respect to this basis, i.e.

$$(T\alpha)_i = \sum_{j=1}^n T_{ij} \alpha_j \quad i = 1, \dots, n,$$

then T is positive if and only if $T_{ij} \geq 0$ for all $i, j = 1, \dots, n$.

¹Whenever scalar multiplication for complex numbers is defined, then linear actually means complex linear.

Remark 2.1.3 If μ is σ -finite and T is given by a measurable kernel $k : \Omega \times \Omega \rightarrow \mathbb{C}$, i.e. if for μ almost all $\omega \in \Omega$

$$Tf(\omega) = \int_{\Omega} k(\omega, \omega') f(\omega') d\mu(\omega'),$$

then $T \geq 0$ if $k \geq 0$ $\mu \times \mu$ almost everywhere.

Examples of this type:

Convolution on $L^p(\mathbb{R}, \lambda)$ with Gauss, Poisson or other kernels.

For later use we note:

Proposition 2.1.2 *If $T : L^p(\Omega, \mu; \mathbb{C}) \rightarrow L^p(\Omega, \mu; \mathbb{C})$ is positive, then*

$$\|T\| = \sup \{ \|Tf\| : f \geq 0, \|f\| = 1 \}.$$

Proof: It is obvious that the above right hand side is dominated by $\|T\|$. Since $T = (T_{\mathbb{R}})_{\mathbb{C}}$, we only need to prove

$$\|T_{\mathbb{R}}\| \leq \sup \{ \|Tf\| : f \geq 0, \|f\| = 1 \} =: \lambda.$$

But if $f \in L^p(\Omega, \mu; \mathbb{R})$, then, denoting $f^+ = f \vee 0$, $f^- = (-f) \vee 0$,

$$Tf = Tf^+ - Tf^-$$

and

$$\begin{aligned} |Tf| &\leq |Tf^+| + |Tf^-| \\ &= Tf^+ + Tf^- \\ &= T(f^+ + f^-). \end{aligned}$$

We obtain

$$\begin{aligned} \|Tf\| &\leq \|T(f^+ + f^-)\| \\ &\leq \lambda \|f^+ + f^-\| = \lambda \|f\|, \end{aligned}$$

since $\mu(\text{supp } f^+ \cap \text{supp } f^-) = 0$. □

2.2 The Dilation Theorem of Akçoglu and Sucheston and its Proof in the finite dimensional Case

The results and proofs presented in this chapter are due to Akçoglu and Sucheston. We adapted them from the publications [1] and [2].

Theorem 2.2.1 (Akçoglu and Sucheston[2]) *Assume $1 \leq p < \infty$ and let $E = L^p(\Omega, \mu)$ be an \mathcal{L}^p -space, $T : E \rightarrow E$ a positive contraction. Then there exist another \mathcal{L}^p -space $\tilde{E} = L^p(\Omega', \mu')$ together with a positive invertible isometry $S : \tilde{E} \rightarrow \tilde{E}$, such that*

$$DT^k = PS^kD \quad \text{for } k = 0, 1, 2, \dots,$$

for some positive isometric embedding $D : E \rightarrow \tilde{E}$ and a norm non-increasing positive projection $P : \tilde{E} \rightarrow \tilde{E}$.

We need prove this only for real scalars, and we first assume that E is finite dimensional. Then for some $\omega_1 > 0, \dots, \omega_n > 0$

$$E = l_n^p(\omega) = \{(\alpha_1, \dots, \alpha_n) : \alpha_i \in \mathbb{R}\} \quad \text{and} \quad \|\alpha\|_E = \left(\sum_{i=1}^n |\alpha_i|^p \omega_i \right)^{\frac{1}{p}}.$$

Moreover, E is isometric to l_n^p , by multiplication with $\omega_i^{\frac{1}{p}}$:

$$\begin{aligned} m_\omega : (\alpha_1, \dots, \alpha_n) &\rightarrow (\alpha_1 \omega_1^{\frac{1}{p}}, \dots, \alpha_n \omega_n^{\frac{1}{p}}) \\ m_\omega : l_n^p(\omega) &\rightarrow l_n^p. \end{aligned}$$

Thus we may assume that $E = l_n^p$, for otherwise it would be sufficient to argue for $T' = m_\omega T m_\omega^{-1}$, $T' : l_n^p \rightarrow l_n^p$.

Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$, and let $T^* : l_n^q \rightarrow l_n^q$ denote the adjoint to T so that for $\alpha \in l_n^p = E$, $\beta \in l_n^q = E^*$

$$(T\alpha, \beta) = (\alpha, T^*\beta),$$

where the bilinear pairing $(\cdot, \cdot) : E \times E^* \rightarrow \mathbb{C}$ is given by

$$(\alpha, \beta) = \sum_{i=1}^n \alpha_i \beta_i.$$

For $1 < p < \infty$ we define a mapping

$$\begin{aligned} * & : E^+ \rightarrow E^{*+} \\ (\alpha_1, \dots, \alpha_n) &\mapsto (\alpha_1^{p-1}, \dots, \alpha_n^{p-1}), \end{aligned}$$

such that

$$\begin{aligned} \|\alpha^*\|_q^q &= \sum_{i=1}^n |\alpha_i^{p-1}|^q = \sum_{i=1}^n \alpha_i^{pq-q} = \\ &= \sum_{i=1}^n \alpha_i^p = \|\alpha\|_p^p. \end{aligned}$$

(If $p = 1$, then let $\alpha_i^* = 1$ if $\alpha_i \neq 0$, $\alpha_i^* = 0$ if $\alpha_i = 0$.)

We further define

$$\begin{aligned} M : E^+ &\rightarrow (E^*)^+ \\ \text{by } M\alpha &= T^*(T\alpha)^*. \end{aligned}$$

Lemma 2.2.1 *Assume $1 < p < \infty$. For $\alpha \in E^+$ with $\|T\alpha\|_p = \|T\| \|\alpha\|_p$ there holds true:*

$$M\alpha = \|T\|^p \alpha^{p-1}.$$

Proof: We have

$$\|M\alpha\|_q \leq \|T^*\| \|(T\alpha)^*\|_q = \|T\| \|T\alpha\|_p^{p-1} = \|T\|^p \|\alpha\|_p^{p-1},$$

and if q is defined by $\frac{1}{p} + \frac{1}{q} = 1$:

$$\|\alpha\|_p \|M\alpha\|_q \geq (\alpha, M\alpha) = (T\alpha, (T\alpha)^{p-1}) = \|T\alpha\|^p = \|T\|^p \|\alpha\|_p^p.$$

Hence there is equality in Hölders inequality. It follows that for some $\lambda \geq 0$ $M\alpha = \lambda \alpha^{p-1}$. Then $(\alpha, \lambda \alpha^{p-1}) = \|T\|^p \|\alpha\|_p^p$ shows $\lambda = \|T\|^p$. \square

Let $E(X) = \{\alpha \in E : \text{supp } \alpha := \{i \in \{1, \dots, n\} : \alpha_i \neq 0\} \subset X\}$ and define $T_X \alpha = T(\chi_X \cdot \alpha)$. We denote $\lambda_X = \sup\{\|T\alpha\| : \alpha \in E(X)^+, \|\alpha\| = 1\}$ the norm of T_X .

Corollary 2.2.1 *Assume $1 < p < \infty$. If $\alpha \in E(X)^+$ with $\|T\alpha\| = \lambda_X \|\alpha\|$, then*

$$\chi_X \cdot M\alpha = \lambda_X^p \alpha^{p-1}.$$

Proof: This follows immediately from the above lemma, applied to the operator T_X acting on the space $E(X)$:

In fact, from

$$\chi_X \cdot T^* = (T_X)^*$$

we infer

$$\begin{aligned} \chi_X \cdot M\alpha &= \chi_X \cdot T^*((T\alpha)^{p-1}) = (T_X)^*((T\alpha)^{p-1}) \\ &= (T_X)^*((T_X\alpha)^{p-1}) = \lambda_X^p \alpha^{p-1}. \end{aligned}$$

□

Lemma 2.2.2 *Assume $1 < p < \infty$. Let $\alpha, \beta \in E^+$ fulfil $\alpha \cdot \beta = 0$ and $M\alpha \leq \alpha^{p-1}$. Then $\alpha M\beta = 0$ and $M(\alpha + \beta) = M\alpha + M\beta$.*

Proof: We have $0 \leq (T\beta, (T\alpha)^{p-1}) = (\beta, M\alpha) \leq (\beta, \alpha^{p-1}) = 0$, since $\alpha \cdot \beta = 0$. Since T is a positive operator, $T\beta$ and $T\alpha$ must have disjoint supports. Hence $0 = (T\alpha, (T\beta)^{p-1}) = (\alpha, M\beta)$, from which we infer $\alpha \cdot M\beta = 0$.

Again the disjointness of the supports of $T\alpha$ and $T\beta$ implies $(T(\alpha + \beta))^{p-1} = (T\alpha + T\beta)^{p-1} = (T\alpha)^{p-1} + (T\beta)^{p-1}$. Applying T^* to this equality finally shows $M(\alpha + \beta) = M\alpha + M\beta$. □

Lemma 2.2.3 *Assume $1 < p < \infty$. Let $\alpha \in E^+$ satisfy $M\alpha \leq \alpha^{p-1}$, and assume that some coordinates of α vanish. Then there exists $\tilde{\alpha}$ with strictly larger support than α , such that*

$$M\tilde{\alpha} \leq \tilde{\alpha}^{p-1}.$$

Proof: Let $X = \{i : \alpha_i = 0\}$. Since E is finite dimensional there exists, by compactness of $\{r \in E(X) : \|r\| = 1\}$, some $\beta \in E(X)$ with $\|\beta\| = 1$ such that

$$\|T\beta\| = \|T_X\beta\| = \|T_X\| \|\beta\|.$$

It follows that

$$\chi_X M\beta \leq \beta^{p-1}.$$

Then, since $\alpha \cdot \beta = 0$, we may apply the last lemma to the result that $\alpha M\beta = 0$, and hence $\text{supp } M\beta \subset X$.

Now we obtain

$$\begin{aligned} M(\alpha + \beta) &= M\alpha + M\beta = M\alpha + \chi_X M\beta \\ &\leq \alpha^{p-1} + \beta^{p-1} = (\alpha + \beta)^{p-1}, \end{aligned}$$

and thus $\tilde{\alpha} = \alpha + \beta$ can be chosen. □

Theorem 2.2.2 *There exists $u \in E^+$ with strictly positive coordinates such that*

$$Mu \leq u^{p-1}.$$

Proof: If $p = 1$ take $u \equiv 1$, i.e. $u_i = 1$ for $i = 1, \dots, n$.
 Otherwise this follows immediately by Lemma 2.2.3. \square

Remark 2.2.1 Assume $p > 1$ and $\|T\| = 1$.

- (i) For $u \in E^+$ with $\|u\|_p = 1$ the assertions $Mu = u^*$ and $\|Tu\|_p = 1$ are equivalent. In fact, denoting $v = Tu$,

$$\begin{aligned} \|Tu\|_p &= (v, v^*) = (Tu, v^*) \\ &= (u, T^*v^*) = (u, Mu). \end{aligned}$$

Hence $\|Tu\|_p = 1$ implies, by the converse to Hölder's inequality, that $Mu = u^* = u^{p-1}$. The converse is evident.

From Proposition 2.1.2 we know that u can be chosen in E_+ . In the case that $\|T\| = 1$ the proof of Theorem 2.2.2 is thus much easier.

- (ii) Assume further all entries of the matrix of T to be strictly positive. Then, if $u \in E^+$ and if $\|Tu\|_p = 1 = \|u\|_p$, the vectors

$$v = Tu \quad \text{and} \quad u^* = T^*v$$

have strictly positive coordinates. This can be seen from the equalities

$$v_j = \sum_{i=1}^n T_{ji}u_i, \quad u_i^{p-1} = \sum_{j=1}^n T_{ji}v_j^{p-1}.$$

For $E = l_n^p$ the measure space Ω' appearing in the statement of the dilation theorem, will be a subset of \mathbb{R}^2 , μ' will be the restriction of the 2-dimensional Lebesgue measure to Ω' and S will be constructed from a point transformation $\tau : \Omega' \rightarrow \Omega'$ taking into account the Radon-Nikodym derivative of $\mu' \circ \tau^{-1}$ with respect to μ' .

For $i = 1, \dots, n$ let I_i be pairwise disjoint intervals on the x -axes of \mathbb{R}^2 of length $l(I_i) = 1$ each. Let J_i , $i = 1, \dots, n$ be mutually disjoint intervals too, again each of length one but on the y -axes.

Set

$$X_i = I_i \times J_i, \quad i = 1, \dots, n; \quad Z_0 = \bigcup_{i=1}^n X_i.$$

For $k \neq 0$, $k \in \mathbb{Z}$ let Z_k be mutually disjoint rectangles, each disjoint from Z_0 too, of positive finite 2-dimensional Lebesgue measure.

By Theorem 2.2.2 there is $u = (u_1, \dots, u_n) \in E^+$, $u_i > 0$ for all $i \in \{1, \dots, n\}$, with

$$Mu \leq u^{p-1}.$$

Let

$$v = (v_1, \dots, v_n) = Tu$$

be its image under T . (It well might happen that $v_j = 0$ for some j .) Let $I = \{1, \dots, n\}$ and $J = \{j \in I : v_j \neq 0\}$.

Define $P = I \times J$ as an index set,

$$\begin{aligned}\xi_{ij} &:= T_{ji} \frac{u_i}{v_j}, & (i, j) \in P \\ \eta_{ij} &:= T_{ji} \left(\frac{v_j}{u_i} \right)^{p-1}, & (i, j) \in I \times I.\end{aligned}$$

Since

$$\begin{aligned}v_j = (Tu)_j &= \sum_{i=1}^n T_{ji} u_i \quad \text{holds true, we have} \\ \sum_{i=1}^n \xi_{ij} &= 1 \quad \text{for all } j \in J.\end{aligned}$$

Similarly, from

$$\begin{aligned}T^*(Tu)^{p-1} &= Mu \leq u^{p-1}, \quad \text{we obtain} \\ \sum_{i=j}^n \eta_{ij} &\leq 1 \quad \text{for all } i \in I.\end{aligned}$$

Divide each I_j , $j \in J$, in n subintervals I_{ij} with length ξ_{ij} , and for each $i \in I$ choose n subintervals J_{ij} in J_i with length η_{ij} . It well might happen, that some of those intervals degenerate, e.g. if $(i, j) \notin P$ then $\eta_{ij} = 0$.

Let for $(i, j) \in P$

$$\begin{aligned}S_{ij} &= I_{ij} \times J_j, \\ R_{ij} &= I_i \times J_{ij},\end{aligned}$$

and define

$$S = \bigcup_{(i,j) \in P} S_{ij}, \quad R = \bigcup_{(i,j) \in P} R_{ij}.$$

Now there are affine transformations

$$\begin{aligned}\tau_{ij} : R_{ij} &\rightarrow S_{ij}, \\ \tau_{ij}(x, y) &= (a_{ij}x + b_{ij}, c_{ij}y + d_{ij}) \quad (x, y) \in R_{ij}\end{aligned}$$

for some $a_{ij}, b_{ij}, c_{ij}, d_{ij} \in \mathbb{R}$, which are surjective up to sets of Lebesgue measure zero.

We are going to define a point transformation τ of $\bigcup_{k=-1}^{-\infty} Z_k \cup Z_0 \cup \bigcup_{k=1}^{+\infty} Z_k$ onto itself:

1. If $R = Z_0$ then define τ as the identity on $\bigcup_{k=1}^{+\infty} Z_k$ otherwise (piecewise affine) to transport $Z_0 \setminus R$ onto Z_1 , Z_k onto Z_{k+1} $k \geq 1$.
2. If $S = Z_0$ let τ be the identity on $\bigcup_{k=-1}^{-\infty} Z_k$ otherwise let τ map Z_k onto Z_{k+1} $k \leq -2$ and Z_{-1} onto $Z_0 \setminus S$.

3. Now it only remains to define

$$\tau : R \rightarrow S$$

by

$$\tau|_{R_{ij}} = \tau_{ij}.$$

The Figure 2.1 indicates the properties of the point transformation τ in an example for the dimension $n = 4$.

Let $\Omega' := \bigcup_{k \in \mathbb{Z}} Z_k$ and let μ' be the restriction of the two dimensional Lebesgue measure to Ω' . Define $\nu = \mu' \circ \tau^{-1}$ and denote ρ the Radon Nikodym derivative of ν with respect to μ' .

For $(i, j) \in P$, $(x, y) \in S_{ij}$

$$\begin{aligned} \rho(x, y) &= \frac{d\nu}{d\mu'}(x, y) = \frac{\mu'(\tau^{-1}(S_{ij}))}{\mu'(S_{ij})} = \frac{\mu'(R_{ij})}{\mu'(S_{ij})} = \frac{\eta_{ij}}{\xi_{ij}} \\ &= \frac{T_{ji}(\frac{v_j}{u_j})^{p-1}}{T_{ji}(\frac{u_i}{v_j})} = \left(\frac{v_j}{u_i}\right)^p (=:\rho_{ij}). \end{aligned}$$

1. Then $S : f \mapsto Sf$ defined by

$$Sf(x, y) := \rho(x, y)^{\frac{1}{p}} f(\tau^{-1}(x, y)), \quad (x, y) \in \Omega',$$

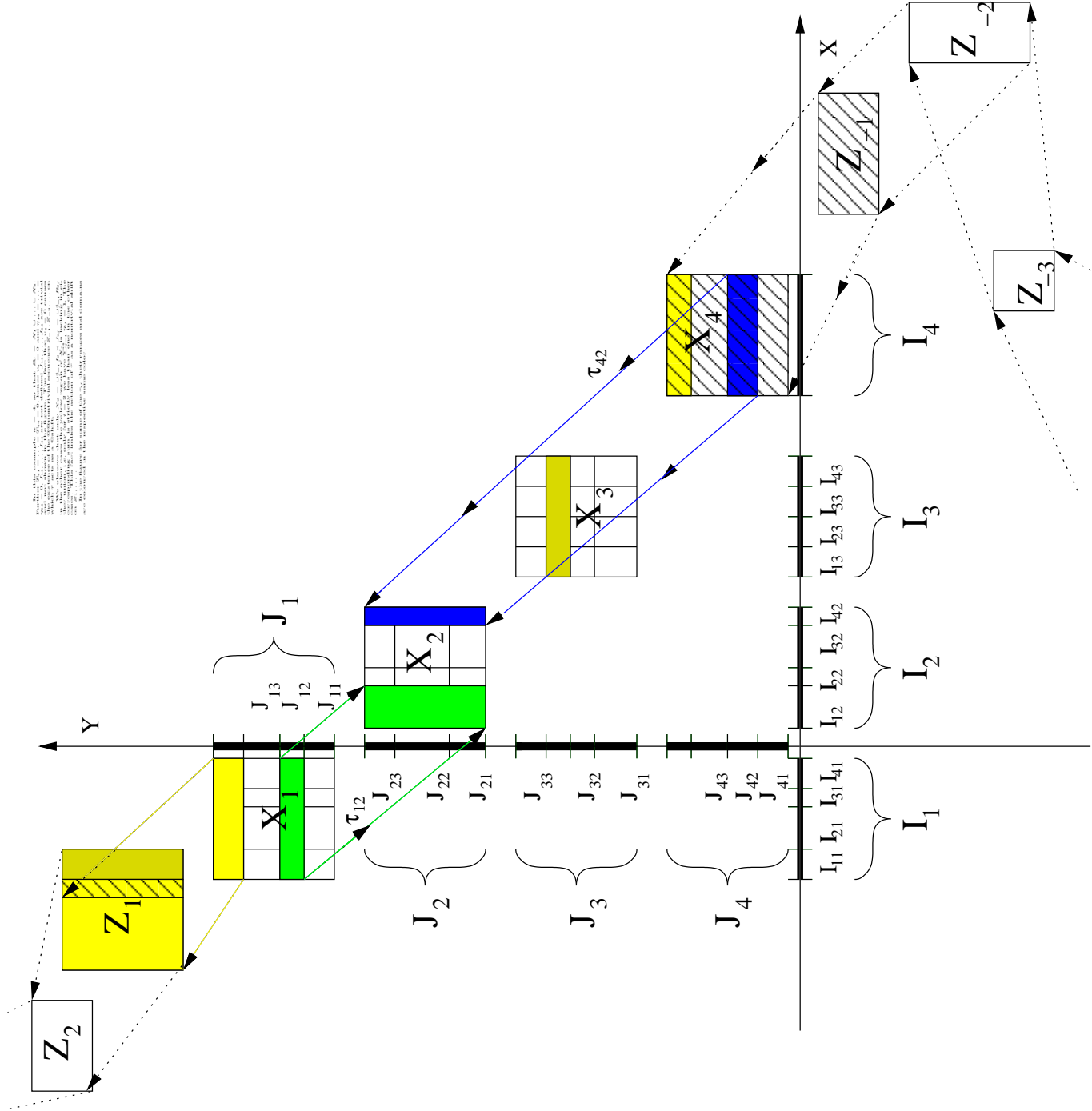
is an invertible isometry of $L^p(\Omega', \mu') = \tilde{E}$.

2. $P : \tilde{E} \rightarrow \tilde{E}$ is just averaging over the sets X_i :

$$Pf(x, y) = \begin{cases} \frac{1}{\mu'(X_i)} \int_{X_i} f d\mu' & \text{whenever } (x, y) \in X_i \text{ for some } i \in I \\ 0 & \text{if } (x, y) \in Z_k \text{ for } k \neq 0. \end{cases}$$

3. $D : E \rightarrow \tilde{E}$ is given by

$$Df(\alpha_1, \dots, \alpha_n) = \sum_{i \in I} \alpha_i \chi_{X_i}(x, y), \quad (x, y) \in \Omega'.$$

Figure 2.1: The Transformation τ .

Lemma 2.2.4 *If $f, g \in E'$ are two functions supported in $Z_0 \cup \bigcup_{k \geq 1} Z_k$ which depend only on the x -coordinate, then $Pf = Pg$ implies $PSf = PSg$. Furthermore, PSf and PSg have their supports in $Z_0 \cup \bigcup_{k \geq 1} Z_k$ again.*

Proof: If $f : \Omega' \rightarrow \mathbb{R}$ depends only on the x -coordinate, then, for some function $F : \mathbb{R} \rightarrow \mathbb{R}$, we have $f(x, y) = F(x)$, $(x, y) \in \Omega'$.

For $j \in J$ we compute

$$\begin{aligned}
 \int_{X_j} Sf(x, y) d\mu'(x, y) &= \sum_{i \in I} \int_{S_{ij}} \frac{v_j}{u_i} f(\tau_{ij}^{-1}(x, y)) dx dy \\
 &= \sum_{i \in I} \frac{v_j}{u_i} \frac{\mu'(S_{ij})}{\mu'(R_{ij})} \int_{R_{ij}} f(x_{ij}) dx dy \\
 &= \sum_{i \in I} \left(\frac{v_j}{u_i} \right)^{1-p} \int_{R_{ij}} F(x) dx dy \\
 &= \sum_{i \in I} \left(\frac{v_j}{u_i} \right)^{1-p} \eta_{ij} \int_{I_i} F(x) dx \\
 &= \sum_{i \in I} T_{ji} \int_{X_i} f(x, y) d\mu'(x, y).
 \end{aligned}$$

If $j \in I \setminus J$ then $X_j \subset Z_0 \setminus S$ and $\tau^{-1}(X_j) \subset Z_{-1}$. It follows that $PSf = 0$ on X_j whenever $f = 0$ on Z_{-1} . Hence,

$$PSf = \sum_{j \in J} \sum_{i \in I} T_{ji} \int_{X_i} f(x, y) d\mu'(x, y) \chi_{X_j} \quad (2.1)$$

$$= \sum_{j \in I} \sum_{i \in I} T_{ji} \int_{X_i} f(x, y) d\mu'(x, y) \chi_{X_j}. \quad (2.2)$$

Here we could extend the summation with respect to j from J to all of $I = \{1, \dots, n\}$, since for $j_0 \in I \setminus J$ we have $T_{j_0 i} = 0 \quad \forall i \in I$. In fact, because of $u_i > 0 \quad \forall i \in I$, it follows that $j_0 \in I \setminus J$, i.e. $0 = v_{j_0} = \sum_{i \in I} T_{j_0 i} u_i$, implies $T_{j_0 i} = 0 \quad \forall i \in I$.

If $Pf = Pg$ and f, g are both supported in $Z_0 \cup \bigcup_{k \geq 1} Z_k$, then for all $i \in I$ $\int_{X_i} f(x, y) d\mu'(x, y) = \int_{X_i} g(x, y) d\mu'(x, y)$, and hence

$$PSf = PSg.$$

□

Further, for $\alpha \in E$ we may restate (2.1) as:

Lemma 2.2.5 $PS(\sum_{i \in I} \alpha_i \chi_{X_i}) = \sum_{j \in J} (T\alpha)_j \chi_{X_j}.$

Theorem 2.2.3 *If $k \in \mathbb{Z}_+$, $\alpha \in E$, then*

$$PS^k \sum_{i \in I} \alpha_i \chi_{X_i} = \sum_{j \in J} (T^k \alpha)_j \chi_{X_j}.$$

Proof: We prove the theorem by induction on k . The assertion is clear for $k = 0$ and just proved and stated in the last lemma for $k = 1$. If $f \in \tilde{E}$ depends only on the x -coordinate, then the same is true for Sf , since τ is piecewise affine and also the Radon Nikodym derivative depends only on the x -coordinate. Hence $S^k f$ depends only on the x -coordinate. Then from Lemma 2.2.4

$$\begin{aligned} PS^{k+1} \sum \alpha_i \chi_{X_i} &= PS P S^k \sum \alpha_i \chi_{X_i} \\ &= PS \left(\sum_{i \in I} (T^k \alpha)_i \chi_{X_i} \right) \\ &= \sum_{j \in J} (T^{k+1} \alpha)_j \chi_{X_j}. \end{aligned}$$

This proves the Akçoglu-Sucheston theorem in the finite dimensional case.

□

Chapter 3

Ultraproducts of Banach Spaces

3.1 The general Banach space Ultraproduct Construction

This construction might be based either on the notion of an ultrafilter or equivalently (at least in the case we are interested in) on points of the Gelfand space of the algebras l^∞ .

Let (A, \leq) be a partially ordered, directed set and let \bar{A} denote the Gelfand space of $l^\infty(A; \mathbb{C})$, i.e. the set of multiplicative functionals $\alpha : l^\infty(A; \mathbb{C}) \rightarrow \mathbb{C}$ with the weak- $*$ -topology. Then $A \hookrightarrow \bar{A}$ is canonically embedded in the compact Hausdorff topological space \bar{A} and is a dense subset in there.

For $f \in l^\infty(A; \mathbb{R})$ define

$$\liminf_{\alpha \in A} f(\alpha) = \sup_{\beta \in A} \inf_{\alpha \geq \beta} f(\alpha)$$

and

$$\limsup_{\alpha \in A} f(\alpha) = \inf_{\beta \in A} \sup_{\alpha \geq \beta} f(\alpha).$$

Proposition 3.1.1 *There exists a point $\text{LIM} \in \bar{A}$ such that*

$$\liminf_{\alpha \in A} f(\alpha) \leq \text{LIM}(f) \leq \limsup_{\alpha \in A} f(\alpha) \quad \forall f \in l^\infty(A; \mathbb{R}).$$

Proof: For $\alpha \in A$ let $A_\alpha = \{\beta : \beta \geq \alpha\}$ and let \bar{A}_α be its closure in \bar{A} . Since A is a directed set, there exists, for $\alpha, \beta \in A$, some $\gamma \in A$ such that

$$\emptyset \neq A_\gamma \subset A_\alpha \cap A_\beta.$$

Hence the sets $(\overline{A_\alpha})_{\alpha \in A}$ have the finite intersection property, i.e. for any finitely many $\alpha_1, \dots, \alpha_n : \bigcap_{i=1}^n \overline{A_{\alpha_i}} \neq \emptyset$. In the compact space \overline{A} there then exists a point

$$\text{LIM} \in \bigcap_{\alpha \in A} \overline{A_\alpha}.$$

But this implies that for any $f \in l^\infty(A; \mathbb{R})$

$$\text{LIM}(f) \in \bigcap_{\alpha \in A} f(\overline{A_\alpha}) \subseteq \bigcap_{\alpha \in A} \overline{f(A_\alpha)},$$

and the last set is contained in the interval

$$\left[\liminf_{\alpha \in A} f(\alpha), \limsup_{\alpha \in A} f(\alpha) \right],$$

which proves the proposition. □

Remark 3.1.1

(i) If $f \in l^\infty(A; \mathbb{C})$ and $u : \overline{f(A)} \rightarrow C$ is continuous, then

$$\text{LIM } u \circ f = u(\text{LIM } f).$$

In fact if \hat{f} denotes the Gelfand transform of f , then $u \circ f = (u \circ \hat{f})|_A$ and

$$\begin{aligned} \text{LIM } u \circ f &= (u \circ \hat{f})(\text{LIM}) = u \circ \hat{f}(\text{LIM}) \\ &= u(\hat{f}(\text{LIM})) = u(\text{LIM } f). \end{aligned}$$

(ii) If $f, g \in l^\infty(A; \mathbb{R})$ are such that $f \leq g$, then

$$\text{LIM } f \leq \text{LIM } g.$$

Clearly $f \leq g$ on A implies $\hat{f} \leq \hat{g}$ on \overline{A} .

(iii) LIM is usually called a generalised limit, or Banach limit. Its value at a function $f \in l^\infty(A; \mathbb{C})$ we sometimes denote $\text{LIM}_{\alpha \in A} f(\alpha)$.

Now let (A, \leq, LIM) be as above. For a net $(E_\alpha, \|\cdot\|_\alpha)_{\alpha \in A}$ of Banach spaces let $\Lambda(A, E)$ denote the space of those functions $f : A \rightarrow \bigcup_{\alpha \in A} E_\alpha$ such that

$$f(\alpha) \in E_\alpha \quad \forall \alpha \in A \quad \text{and} \quad \|f\|_\infty = \sup_\alpha \|f(\alpha)\|_\alpha < \infty.$$

Then clearly $\Lambda(A, E), \|\cdot\|_\infty$ is a normed vector space, it is even complete and a Banach lattice if all E_α are Banach lattices. The operations are defined pointwise:

$$\begin{aligned} (f+g)(\alpha) &= f(\alpha) + g(\alpha), \quad \alpha \in A \\ f \wedge g(\alpha) &= f(\alpha) \wedge g(\alpha), \quad \alpha \in A \\ \text{etc.} \end{aligned}$$

Let us define a semi-norm $\|\cdot\|_{\text{LIM}}$ on $\Lambda(A, E)$ by

$$\|f\|_{\text{LIM}} = \text{LIM}_{\alpha \in A} \|f(\alpha)\|_\alpha.$$

We let $N = \{f \in \Lambda(A, E) : \|f\|_{\text{LIM}} = 0\}$ denote its kernel and denote

$$\prod_{\text{LIM}} E_\alpha = \Lambda(A, E)/N.$$

Further, for $f, g \in \Lambda(A, E)$, we write $f \sim g$ if $f - g \in N$ and $[f] = f + N$.

The following useful observation seems to be due to Akçoglu and Sucheston [2].

Proposition 3.1.2 $\Lambda(A, E)/N$ is a Banach space.

This proposition will be proved by the following two lemmata.

Lemma 3.1.1 For $f \in \Lambda(A, E)$ there exists $g \in \Lambda(A, E)$ such that

$$f \sim g \text{ and } \|g(\alpha)\|_\alpha \leq \|f\|_{\text{LIM}} \quad \forall \alpha \in A.$$

Proof: For $\alpha \in A$ denote $\lambda_\alpha = \|f\|_{\text{LIM}} / \max\{\|f(\alpha)\|_\alpha, \|f\|_{\text{LIM}}\}$ and define $g \in \Lambda(A, E)$ by

$$g(\alpha) = \lambda_\alpha f(\alpha), \quad \alpha \in A.$$

Then,

$$\begin{aligned} \|f(\alpha) - g(\alpha)\|_\alpha &= \|(1 - \lambda_\alpha)f(\alpha)\|_\alpha \\ &= (1 - \lambda_\alpha) \|f(\alpha)\|_\alpha. \end{aligned}$$

Hence,

$$\begin{aligned} \|f - g\|_{\text{LIM}} &\leq \text{LIM}_{\alpha \in A} (1 - \lambda_\alpha) \cdot \text{LIM}_{\alpha \in A} \|f(\alpha)\|_\alpha \\ &= 0 \cdot \|f\|_{\text{LIM}}. \end{aligned}$$

It is clear that for all $\alpha \in A$: $\|g(\alpha)\|_\alpha = \lambda_\alpha \|f(\alpha)\|_\alpha \leq \|f\|_{\text{LIM}}$. □

Now the completeness of $\Lambda(A, E)/N$ follows from:

Lemma 3.1.2 *Let $(f_n)_{n=1}^\infty \in \Lambda(A, E)$ be a sequence such that $\sum_{n=1}^\infty \|f_n\|_{\text{LIM}} < \infty$. Then there exists*

$$f \in \Lambda(A, E) \quad \text{such that} \quad \lim_{N \rightarrow \infty} \left\| f - \sum_{n=1}^N f_n \right\|_{\text{LIM}} = 0.$$

Proof: Let $g_n \sim f_n$ be such that $\|g_n(\alpha)\|_\alpha \leq \|f_n\|_{\text{LIM}}$ for all $\alpha \in A$. Then $g(\alpha) = \sum_{n=1}^\infty g_n(\alpha) \in E_\alpha$ exists, since $\sum_{n=1}^\infty \|g_n(\alpha)\|_\alpha \leq \sum_{n=1}^\infty \|f_n\|_{\text{LIM}} < \infty$. This estimate further shows $g \in \Lambda(A, E)$. It is clear that for all $N \in \mathbb{N}$: $\sum_{n=1}^N g_n \sim \sum_{n=1}^N f_n$.
Now,

$$\begin{aligned} \left\| g - \sum_{n=1}^N f_n \right\|_{\text{LIM}} &= \left\| g - \sum_{n=1}^N g_n \right\|_{\text{LIM}} = \\ &= \text{LIM}_{\alpha \in A} \left\| \sum_{n=N+1}^\infty g_n(\alpha) \right\|_\alpha \leq \text{LIM}_{\alpha \in A} \sum_{n=N+1}^\infty \|g_n(\alpha)\|_\alpha \\ &\leq \sum_{n=N+1}^\infty \|f_n\|_{\text{LIM}} \rightarrow 0 \text{ if } N \rightarrow \infty. \quad \square \end{aligned}$$

Remark 3.1.2

- (i) If $E = (E_\alpha)_{\alpha \in A}$, $F = (F_\alpha)_{\alpha \in A}$ are as above and $u_\alpha : E_\alpha \rightarrow F_\alpha$ is a bounded linear operator such that $\sup_{\alpha \in A} \|u_\alpha\|_\alpha < \infty$, then there exists a linear operator

$$u : \Lambda(A, E) \rightarrow \Lambda(A, F)$$

defined by

$$u(f_\alpha) = u_\alpha(f_\alpha), \quad \alpha \in A.$$

Furthermore,

$$\|u(f)\|_{\text{LIM}} \leq \sup_{\alpha \in A} \|u_\alpha\|_\alpha \|f\|_{\text{LIM}}.$$

Hence u passes to the quotient spaces and defines a bounded linear map, the ultraproduct of the net $(u_\alpha)_{\alpha \in A}$, denoted

$$\prod_{\text{LIM}} u_\alpha : \prod_{\text{LIM}} E \rightarrow \prod_{\text{LIM}} F,$$

of norm at most $\sup_{\alpha \in A} \|u_\alpha\|$.

- (ii) If E is a Banach space and $(E_\alpha)_{\alpha \in A}$ is a net of its subspaces ($E_\alpha \subseteq E \quad \forall \alpha \in A$), ordered by inclusion (i.e. $\alpha' \leq \alpha \Leftrightarrow E_{\alpha'} \subseteq E_\alpha$), such that $\bigcup_{\alpha \in A} E_\alpha$ is dense in E , then E is canonically embedded in $\prod_{\text{LIM}} E_\alpha$ by

just taking the ultraproduct of the inclusions.

For, if $f \in E$, then there exists a net $f_\alpha \in E_\alpha$ such that $\|f - f_\alpha\|_E \rightarrow 0$.

Define $\tilde{f} \in \prod_{\text{LIM}} E_\alpha$ as $\tilde{f} = [f_\alpha]$, then

$$\|\tilde{f}\| = \text{LIM}_{\alpha \in A} \|f_\alpha\|_\alpha = \text{LIM}_{\alpha \in A} \|f_\alpha\|_E = \lim_{\alpha \rightarrow \infty} \|f_\alpha\| = \|f\|.$$

It is straightforward to check, that the definition of \tilde{f} does not depend on the special net $(f_\alpha)_{\alpha \in A}$ and that $f \mapsto \tilde{f}$ is linear.

- (iii) The (real) $L^p(\Omega, \mu; \mathbb{R})$ spaces, $1 \leq p < \infty$, are characterized by the Kakutani and the Bohnenblust-Nakano theorem (see e.g. [31] Chap.5 §15 Theorem 3) as those Banach lattices E such that

$$f, g \in E^+, f \wedge g = 0 \Rightarrow \|f + g\|^p = \|f\|^p + \|g\|^p.$$

As a corollary to this fact we have:

Corollary 3.1.1 *Let $(E_\alpha)_{\alpha \in A}$ be a net of \mathcal{L}^p -spaces, then there exists a measure space $(\Omega^\circ, \mathfrak{A}^\circ, \mu^\circ)$ such that*

$$\prod_{\text{LIM}} E_\alpha = L^p(\Omega^\circ, \mu^\circ).$$

Proof: Let $f, g \in \prod_{\text{LIM}} E_\alpha^+$ with $f \wedge g = 0$ be represented by $(f_\alpha)_{\alpha \in A}$ respectively $(g_\alpha)_{\alpha \in A}$. Then $h = (f_\alpha \wedge g_\alpha)_{\alpha \in A} \in N$, since it is a representation for $f \wedge g = 0$, and we may further assume that $f_\alpha, g_\alpha \in E_\alpha^+$ for all $\alpha \in A$.

Now, for all $\alpha \in A$, the functions $f_\alpha - (f_\alpha \wedge g_\alpha)$ and $g_\alpha - (f_\alpha \wedge g_\alpha)$ have disjoint supports, hence, using $E_\alpha \in \mathcal{L}^p$,

$$\begin{aligned} & \|f_\alpha - (f_\alpha \wedge g_\alpha) + g_\alpha - (f_\alpha \wedge g_\alpha)\|_{E_\alpha}^p \\ &= \|f_\alpha - (f_\alpha \wedge g_\alpha)\|_{E_\alpha}^p + \|g_\alpha - (f_\alpha \wedge g_\alpha)\|_{E_\alpha}^p. \end{aligned}$$

Because

$$[(f_\alpha - g_\alpha \wedge f_\alpha)_\alpha] = f, \quad [(g_\alpha - (f_\alpha \wedge g_\alpha))_\alpha] = g,$$

we have, by passing to the limit:

$$\begin{aligned} \|f + g\|^p &= \text{LIM}_{\alpha \in A} \|f_\alpha + g_\alpha - 2(f_\alpha \wedge g_\alpha)\|_\alpha^p = \\ &= \text{LIM}_{\alpha \in A} \|f_\alpha - f_\alpha \wedge g_\alpha\|^p + \text{LIM}_{\alpha \in A} \|g_\alpha - (f_\alpha \wedge g_\alpha)\|^p \\ &= \|f\|^p + \|g\|^p. \end{aligned}$$

Remark 3.1.3 A more direct approach to the above Corollary 3.1.1, using directly ultraproducts and not relying on the Bohnenblust-Nakano or on Kakutani's theorem, can be found in the publication [16] of Dacunha Castelle and Krivine.

3.2 The Akçoglu-Sucheston Dilation Theorem

In this section we shall use the foregoing constructions to complete the proof of the Akçoglu-Sucheston dilation theorem in the general case. We shall rather closely follow their arguments.

Definition 3.2.1 *A semi-partition α of Ω is a finite collection of pairwise disjoint subsets (measurable of course)*

$$\alpha = \{X_1, \dots, X_{n_\alpha} : X_i \cap X_j = \emptyset \text{ if } i \neq j\}$$

each one of finite measure.

For a semi-partition α we let $\mathcal{E}_\alpha : E \rightarrow E$ denote the corresponding conditional expectation operator, defined by:

$$\mathcal{E}_\alpha(g)(\omega) = \begin{cases} \frac{1}{\mu(X)} \int_X g(\omega') d\mu(\omega') & \text{if } \omega \in X \in \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.2.1

- (i) The set $A = \{\alpha : \alpha \text{ a semi-partition}\}$ of all semi-partitions is partially ordered, directed by refinement, and for $g \in E$ we have

$$\lim_{\alpha \in A} \|\mathcal{E}_\alpha g - g\|_p \rightarrow 0.$$

- (ii) Further, if $T : E \rightarrow E$ is a positive contraction, and if

$$T_\alpha : E \rightarrow E \text{ is defined by } T_\alpha = \mathcal{E}_\alpha T \mathcal{E}_\alpha,$$

then it may be restricted to $E_\alpha := \mathcal{E}_\alpha E$ and there be viewed as a positive contraction on $E_\alpha \simeq l_{n_\alpha}^p(\omega) \simeq l_{n_\alpha}^p$. (Here \simeq is a positive isometrical isomorphism and ω is the weight sequence $(\mu(X_1), \dots, \mu(X_{n_\alpha}))$.)

- (iii) Since A is directed by refinement and since $\|T\| < \infty$ we obtain by induction from (i):

$$\lim_{\alpha \in A} \|T_\alpha^n f - f\| = 0, \quad n = 0, 1, \dots \quad \forall f \in E.$$

(For a proof just note: $\|\mathcal{E}_\alpha T \mathcal{E}_\alpha f - T f\| \leq \|\mathcal{E}_\alpha T \mathcal{E}_\alpha f - \overbrace{T \mathcal{E}_\alpha f}^{=:g}\| + \|\mathcal{E}_\alpha T \mathcal{E}_\alpha f - T \mathcal{E}_\alpha f\| \leq \|\mathcal{E}_\alpha g - g\| + \|T\| \|\mathcal{E}_\alpha f - f\| \xrightarrow{\alpha \rightarrow \infty} 0$.)

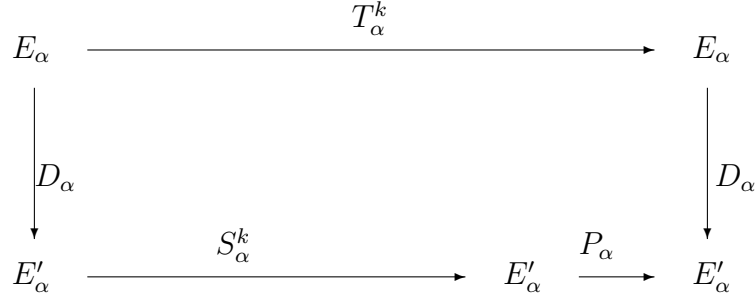


Figure 3.1: Approximating a Dilation

For a semi-partition α we display in Figure 3.1 a dilation of T_α according to what already has been proved (Theorem 2.2.1).

Take a Banach-limit LIM on $l^\infty(A, \mathbb{C})$ such that

$$\liminf f \leq \text{LIM} f \leq \limsup f \quad \forall f \in l^\infty(A, \mathbb{R}).$$

Denote $E' = \Lambda(A, E_\alpha)/N$ and define:

$$\begin{aligned}
D : E &\rightarrow E' & \text{by } f &\mapsto [D_\alpha \mathcal{E}_\alpha f] \\
S : E' &\rightarrow E' & \text{by } Sf &= [S_\alpha f_\alpha] \quad \text{if } f = [f_\alpha] \\
P : E' &\rightarrow E' & \text{by } Pf &= [P_\alpha f_\alpha] \quad \text{if } f = [f_\alpha].
\end{aligned}$$

By what has been proved,

$$P_\alpha S_\alpha^n D_\alpha \mathcal{E}_\alpha f = D_\alpha T_\alpha^n \mathcal{E}_\alpha f = \begin{cases} D_\alpha T_\alpha^n f & n = 1, 2, \dots \\ D_\alpha \mathcal{E}_\alpha f & n = 0 \end{cases} \quad \forall \alpha \in A,$$

and, since D is an isometry and \mathcal{E}_α a norm non-increasing projection:

$$\begin{aligned}
\|D_\alpha T_\alpha^n f - D_\alpha \mathcal{E}_\alpha T_\alpha^n f\| &= \|T_\alpha^n f - \mathcal{E}_\alpha T_\alpha^n f\| \\
&\leq \|T_\alpha^n f - T_\alpha^n f\| \rightarrow 0 \text{ with } \alpha \in A.
\end{aligned}$$

But this shows

$$PS^n Df = DT^n f, \quad n = 1, 2, \dots \quad f \in E,$$

which concludes the proof. □

Remark 3.2.2 Coifman, Rochberg and Weiss [13] extended the result of Akçoglu and Sucheston to cover the case of a sub-positive contraction T (see the pages 58 and 59 of [13]).

For the readers convenience we recall the definition.

For $U : L^p(\Omega, \mu; \mathbb{C}) \rightarrow L^p(\Omega, \mu; \mathbb{C})$, bounded and linear, define $\bar{U} \in B(L^p(\Omega, \mu; \mathbb{C}))$ by

$$\bar{U}f = \overline{(Uf)} \quad f \in L^p(\Omega, \mu; \mathbb{C})$$

($\bar{}$ denoting complex conjugation), and define further $\operatorname{Re} U \in B(L^p(\Omega, \mu; \mathbb{C}))$ by

$$\operatorname{Re} U = \frac{1}{2}(U + \bar{U}).$$

Definition 3.2.2 $T \in B(L^p(\Omega, \mu; \mathbb{C}))$ is called a sub-positive contraction, if there exists a positive contraction R such that $R + \operatorname{Re}(e^{i\theta}T)$ is positive for all $\theta \in [0, 2\pi)$.

Proposition 3.2.1 If $T \in B(l_n^p)$ is represented by its matrix $(T_{ij})_{i,j=1}^n$ with respect to the usual basis of l_n^p , then T is a sub-positive contraction if and only if $(|T_{ij}|)_{i,j=1}^n$ represents a contraction.

Proof: Let R be a contraction such that

$$R + \operatorname{Re}(e^{i\theta}T) \geq 0 \quad \forall \theta.$$

Then

$$R_{ij} + (\operatorname{Re}(e^{i\theta}T))_{ij} = R_{ij} + \operatorname{Re}(e^{i\theta}T)_{ij} \geq 0 \quad \forall i, j, \theta.$$

Hence $R_{ij} - |T_{ij}| \geq 0 \quad \forall i, j$, and by Proposition 2.1.2 it follows that the matrix $(|T_{ij}|)_{i,j=1}^n$ represents a contraction which we denote $|T|$.

The other implication is clear, since we can choose $|T|$ as the positive contraction, named R , in the definition of sub-positivity. □

for a sub-positive contraction T a modification of the proof of the dilation theorem has to be done in the finite dimensional case. For the construction then, Coifman and Weiss use $|T|$ instead of T to obtain the measure space $L^p(\Omega', \mu')$, the affine mappings τ_{ij} and the Radon-Nikodym derivative ρ .

Now the operator S is defined as

$$S f(x, y) = \sum_{i,j} \sigma_{ij} \rho(x, y)^{\frac{1}{p}} f(\tau_{ij}^{-1}(x, y)) \chi_{S_{ij}}(x, y), \quad (x, y) \in \Omega',$$

where $\sigma_{ij} = \operatorname{sign} T_{ij}$.

Definition 3.2.3 We shall say that an operator $R : L^p(\Omega, \mu) \rightarrow L^p(\Omega', \mu')$ preserves separation if $f \cdot g = 0$ implies $Rf \cdot Rg = 0 \quad \forall f, g \in L^p(\Omega, \mu)$, i.e., if the supports of two elements are disjoint, then so are the supports of their images under R .

Remark 3.2.3

- (i) For $1 \leq p < \infty$, $p \neq 2$ it is easy to see that an isometry $S : L^p(\Omega, \mu) \rightarrow L^p(\Omega', \mu')$ is separation preserving.
- (ii) In the case that $p = 2$ the appropriate requirement is that S is a positive isometry.

For the readers convenience we prove the assertions of this remark.

First assume $1 \leq p < \infty$, $p \neq 2$. If $f \cdot g = 0$ then $\|f + g\|^p + \|f - g\|^p = 2(\|f\|^p + \|g\|^p)$ and conversely. Now, if S is an isometry, then this equality holds true for f and g if and only if it holds true for the images Sf and Sg .

Now the first part is proved and to establish the second one we proceed similarly:

For non-negative $f, g \in L^2(\Omega, \mu)$ (resp. $f, g \in L^2(\Omega', \mu')$) we have that $\|f + g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$ if and only if $f \cdot g = 0$. Hence for such functions the same reasoning as above applies.

When f is real valued, then $Sf^+ = S(f^+)$ and $Sf^- = S(f^-)$, since $S(f^+) \cdot S(f^-) = 0$ and $S(f^+), S(f^-) \geq 0$. Now it is easy to see that $\operatorname{Re}(Sf) = S(\operatorname{Re} f)$ and $\operatorname{Im}(Sf) = S(\operatorname{Im} f)$ for a possibly complex valued element $f \in L^2(\Omega, \mu)$.

Since $f \cdot g = 0$ exactly if all products of one element from $\{(\operatorname{Re} f)^+, (\operatorname{Re} f)^-, (\operatorname{Im} f)^+, (\operatorname{Im} f)^-\}$ with an element from $\{(\operatorname{Re} g)^+, (\operatorname{Re} g)^-, (\operatorname{Im} g)^+, (\operatorname{Im} g)^-\}$ vanish one easily concludes using the properties of S established in the last paragraph. □

Lemma 3.2.1 A separation preserving contraction R is a sub-positive contraction.

Proof: For $A \subset \Omega$, $0 < \mu(A) < \infty$ define $h_A(\omega) = R(\chi_A)(\omega)$.

We claim that for $\omega \in \Omega$ and two sets B, C of finite nonzero measure one of the following cases occurs:

1. $h_C(\omega) = 0$ and $h_B(\omega) = 0$

2. $h_C(\omega) = 0$ and $h_B(\omega) \neq 0$
3. $h_C(\omega) \neq 0$ and $h_B(\omega) = 0$
4. $h_C(\omega) = h_B(\omega) \neq 0$

To establish the claim, we may assume that $R\chi_B(\omega) \cdot R\chi_C(\omega) \neq 0$. Since R preserves separation we have $R\chi_{B-B\cap C}(\omega) \cdot R\chi_{C-B\cap C}(\omega) = 0$, and from

$$\begin{aligned} R\chi_{B\cap C}(\omega) &= R\chi_B(\omega) - R\chi_{B-B\cap C}(\omega) \\ &= R\chi_C(\omega) - R\chi_{C-B\cap C}(\omega) \end{aligned}$$

we obtain

$$R\chi_B(\omega) = R\chi_{B\cap C}(\omega) \quad \text{or} \quad R\chi_C(\omega) = R\chi_{B\cap C}(\omega).$$

In the first case $R\chi_B(\omega) - R\chi_{B-B\cap C}(\omega) = R\chi_C(\omega) \neq 0$. Again since R preserves separation, we may use $R\chi_{B-B\cap C}(\omega) \cdot R\chi_C(\omega) = 0$ to obtain $R\chi_{B-B\cap C}(\omega) = 0$, and hence

$$R\chi_B(\omega) = R\chi_C(\omega).$$

The second case yields by symmetry the same result, which establishes the claim.

For $\omega \in \Omega$ we may thus define unambiguously

$$h(\omega) = \begin{cases} h_A(\omega) & \text{if } A \text{ is such that } 0 < \mu(A) < \infty \text{ and } h_A(\omega) \neq 0 \\ 0 & \text{if for all sets of finite measure } A : h_A(\omega) = 0. \end{cases}$$

This function might not be measurable, but for any set B of finite nonzero measure (denote $\tilde{B} = \text{supp} R\chi_B$) the function $h \cdot \chi_{\tilde{B}} = R\chi_B$ is measurable.

If $(A_j)_{j=1}^k$ are pairwise disjoint and $f = \sum \lambda_j \chi_{A_j}$ is a simple function, then

$$Rf = h \sum_{j=1}^n \lambda_j \chi_{\tilde{A}_j}. \quad (3.1)$$

If we define

$$P : L^p(\Omega, \mu) \rightarrow L^p(\Omega, \mu)$$

by

$$Pf(\omega) = |h(\omega)| \sum_{j=1}^n \lambda_j \chi_{\tilde{A}_j}(\omega), \quad (3.2)$$

then we see that for all simple functions f :

$$\begin{aligned} \|Pf\| &= \|Rf\|, \\ Pf + \operatorname{Re} e^{i\theta} Rf &\geq 0 \text{ if } f \geq 0. \end{aligned}$$

Using the boundedness of R and P this extends to all of $L^p(\Omega, \mu)$. \square

Remark 3.2.4 For later use it is worthwhile to note that for a separation preserving contraction R there exists a positive contraction P , such that for all $f \in L^p(\Omega, \mu)$:

$$|Rf|(\omega) = Pf(\omega), \quad \mu\text{-almost everywhere.}$$

This follows from (3.1) and (3.2), whenever $f = \sum \lambda_j \chi_{A_j}$ is a simple function as above, since the sets \tilde{A}_j , $j = 1, \dots, k$ are pairwise disjoint. By continuity it continues to hold for arbitrary $f \in L^p(\Omega, \mu)$.

3.3 A further Ultraproduct Construction

Only later, in section 4.2, we will proof a dilation theorem, see Theorem 4.2.1, for certain continuous semigroups on \mathcal{L}^p -spaces. For its proof we will use once more a Banach space ultraproduct construction, applied to dilations, which are given by the Akçoglu-Sucheston dilation theorem, Theorem 2.2.1. We shall need some considerations on representations of continuous groups on reflexive spaces too, which will be presented in section 4.1, but we found it best to present the ultraproduct part of the proof of Theorem 4.2.1 here.

To fix ideas, in this section we shall consider a continuous one-parameter semigroup $\{T_t : t \geq 0\}$, with $T_0 = \operatorname{id}$, of positive contractions acting on $L^p(\Omega, \mu)$, where $1 \leq p < \infty$. We shall prove a preliminary lemma on the existence of a dilation of the sub-semigroup $\{T_t : t \geq 0, t \in \mathbb{Q}\}$ to a group of isometries of some \mathcal{L}^p -space. (Here \mathbb{Q} denotes the rational numbers.)

Furthermore we remark that, as almost always in this paper, the letters T, S, P, D stand for positive operators between \mathcal{L}^p -spaces, D will denote an isometric embedding, P a contractive projection, T will be a contraction and S an invertible isometry.

Lemma 3.3.1 *If $\{T_t : t \geq 0\}$ is a semigroup in $B(L^p(\Omega, \mu))$, which fulfils the above requirements, then there exists a measure space $(\Omega^\circ, \mathfrak{A}^\circ, \mu^\circ)$ and a group $\{S_s^\circ : s \in \mathbb{Q}\}$ of positive isometries of $L^p(\Omega^\circ, \mu^\circ)$ such that*

$$D^\circ \circ T_s = P^\circ \circ S_s^\circ \circ D^\circ, \quad s \in \mathbb{Q}^+,$$

for a suitable positive isometric embedding $D^\circ : L^p(\Omega, \mu) \rightarrow L^p(\Omega^\circ, \mu^\circ)$ and a positive contractive projection P° acting on $L^p(\Omega^\circ, \mu^\circ)$.

In our paper [19] the proof of this lemma depends on certain filters on the rational numbers. Here instead, we shall use the partial ordering of the natural numbers given by divisibility.

Definition 3.3.1 For $n, m \in \mathbb{N}$ define

$$n \preceq m \text{ if } n \text{ divides } m.$$

Remark 3.3.1

- (i) If we take for $k, l \in \mathbb{N}$ $m = \text{lcm}(k, l)$ to be the least common multiple of k and l , then

$$k \preceq m \text{ and } l \preceq m.$$

Thus (\mathbb{N}, \preceq) is directed.

- (ii) For a finite set $B \subset \mathbb{Q}$ let $U_B := \{n \in \mathbb{N} : ns \in \mathbb{Z} \ \forall s \in B\}$. The latter set just consists of the common multiples of the denominators of the reduced fractions representing the rationals from B .

The fact that the set of all sets $\{U_B : B \subset \mathbb{Q}, \ B \text{ finite}\}$ is closed under finite intersections just corresponds to (\mathbb{N}, \preceq) being directed.

- (iii) We note that the Banach limits given by Proposition 3.1.1 for (\mathbb{N}, \preceq) are in one to one correspondence to the maximal filters containing the above filter-basis.

Proof of the Lemma 3.3.1: According to Theorem 2.2.1, for any $n \in \mathbb{N}$, there exists a dilation of $\{T_{1/n}^k : k \in \mathbb{Z}^+\}$:

$$D_{1/n} \circ T_{1/n}^k = P_{1/n} \circ S_{1/n}^k \circ D_{1/n}, \quad k \in \mathbb{Z}^+,$$

where $D_{1/n}, P_{1/n}, S_{1/n}$ are as above, $S_{1/n}$ acting on some space $L^p(\Omega'_{1/n}, \mu'_{1/n})$.

If we define, for $s \in \mathbb{Q}$, $S_{n,s} \in B(L^p(\Omega'_{1/n}, \mu'_{1/n}))$ by

$$S_{n,s} = \begin{cases} S_{1/n}^{ns} & \text{if } ns \in \mathbb{Z} \\ \text{id} & \text{if } ns \notin \mathbb{Z}, \end{cases}$$

and if $B = \{s_1, \dots, s_k\} \subset \mathbb{Q}^+$ is a finite subset, then for $s \in B$ and $n \in U_B$ the diagram in Figure 3.2 commutes.

Let LIM be a Banach limit for (\mathbb{N}, \preceq) . Then we may form ultraproducts of the spaces and operators involved. From Corollary 3.1.1 we know that there exist measure spaces $(\Omega^\circ, \mathfrak{A}^\circ, \mu^\circ)$ and $(X^\circ, \mathfrak{B}^\circ, \nu^\circ)$ such that

$$\begin{aligned} \prod_{\text{LIM}} L^p(\Omega, \mu) &= L^p(X^\circ, \nu^\circ) \\ \prod_{\text{LIM}} L^p(\Omega'_{1/n}, \mu'_{1/n}) &= L^p(\Omega^\circ, \mu^\circ) \end{aligned}$$

$$\begin{array}{ccccc}
L^p(\Omega, \mu) & \xrightarrow{T_{1/n}^{ns}} & L^p(\Omega, \mu) & & \\
\downarrow D_{1/n} & & \downarrow D_{1/n} & & \\
L^p(\Omega'_{1/n}, \mu'_{1/n}) & \xrightarrow{S_{n,s}} & L^p(\Omega'_{1/n}, \mu'_{1/n}) & \xrightarrow{P_{1/n}} & L^p(\Omega'_{1/n}, \mu'_{1/n}) .
\end{array}$$

Figure 3.2: Dilations for Rationals

can be identified as Banach lattices in each of the two cases.

Let I denote the canonical inclusion $I : L^p(\Omega, \mu) \rightarrow \prod_{\text{LIM}} L^p(\Omega, \mu) = L^p(X^\circ, \nu^\circ)$ and denote

$$\begin{aligned}
D^\circ &= \prod_{\text{LIM}} D_{1/n} \circ I , \\
P^\circ &= \prod_{\text{LIM}} P_{1/n} , \\
S_s^\circ &= \prod_{\text{LIM}} S_{n,s} , \quad s \in \mathbb{Q} .
\end{aligned}$$

Then all the asserted properties of the operators and spaces involved are rather immediate.

To give examples, let us check that

$$S : \mathbb{Q} \rightarrow B \left(\prod_{\text{LIM}} L^p(\Omega'_{1/n}, \mu'_{1/n}) \right) , \quad s \mapsto S_s ,$$

is a group homomorphism, and that we obtained a dilation of the semigroup $\{T_s : s \in \mathbb{Q}^+\}$.

If $s, s' \in \mathbb{Q}$ are given, and if $f \in \prod_{\text{LIM}} L^p(\Omega'_{1/n}, \mu'_{1/n})$ is represented by a sequence $(f_n)_{n \in \mathbb{N}}$, which is possible by Proposition 3.1.2, then for n sufficiently large with respect to our partial ordering \preceq , i.e. if $n \in U_{\{s, s'\}}$, there holds true:

$$\begin{aligned}
S_{n, s+s'}(f_n) &= S_{1/n}^{n(s+s')}(f_n) \\
&= S_{1/n}^{ns}(S_{1/n}^{ns'}(f_n)) \\
&= S_{n,s}(S_{n,s'}(f_n)) .
\end{aligned}$$

By the definition of the partial ordering and by Proposition 3.1.1,

$$\begin{aligned}
&\| (S_{n, s+s'}(f_n))_{n \in \mathbb{N}} - (S_{n,s}(S_{n,s'}(f_n)))_{n \in \mathbb{N}} \|_{\text{LIM}} \leq \\
&\leq \limsup_{n \in \mathbb{N}, \preceq} \| S_{n, s+s'}(f_n) - S_{n,s}(S_{n,s'}(f_n)) \| \\
&= 0 .
\end{aligned}$$

From this we infer that they represent the same elements in $\prod_{\text{LIM}} L^p(\Omega'_{1/n}, \mu'_{1/n})$, and thus

$$S_s^\circ \circ S_{s'}^\circ(f) = S_{s+s'}^\circ(f).$$

Furthermore, for $s \in \mathbb{Q}^+$, the commutativity of the above diagram for large enough n , i.e. for $n \in U_{\{s\}}$, implies, as one can see using a reasoning analogous to the one just given,

$$D^\circ \circ T_s = P^\circ \circ S_s^\circ \circ D^\circ. \quad \underline{\hspace{1cm}} \square$$

Chapter 4

Representations of Commutative topological Groups on reflexive Banach Spaces

4.1 General Remarks

This topic has been discussed in an broader context by I. Glicksberg and K. de Leeuw [22]. Here we shall develop only those parts of their theory which are necessary for our purpose.

Let E be a reflexive Banach space, G a commutative topological group and $\pi : G \rightarrow B(E)$ a uniformly bounded representation of G on E . Thus we are given an algebraic group homomorphism π from G to the invertible elements of $B(E)$ such that $C := \sup_{\pi \in G} \|\pi(x)\| < \infty$.

Our topic here is to study how the subspace

$$E_c = \{\xi \in E : x \mapsto \pi(x)\xi \text{ is continuous from } G \text{ to } E\}$$

of continuously translating elements of E is situated in E .

First, since π is uniformly bounded, E_c is surely a closed subspace of E and it is clearly invariant for $\pi(G)$.

If E were a Hilbert space, then one would take an orthogonal projection P onto E_c . Since G is amenable as a discrete group with an invariant mean on $l^\infty(G)$, *m* say, we then could define a new projection P_G onto E_c in the commutant of $\pi(G)$ by

$$\langle P_G \xi, \eta \rangle = m(x \mapsto \langle \pi(x^{-1})P\pi(x)\xi, \eta \rangle) \quad \forall \xi, \eta \in E.$$

Clearly its norm is bounded by C^2 .

In a reflexive space there is in general no bounded projection onto a closed subspace and we have to use a refinement of the above construction.

Let $\mathcal{U}(e)$ denote the system of open neighbourhoods of the identity $e \in G$ and let, for $U \in \mathcal{U}(e)$,

$$U_\pi = \{ \pi(x) : x \in U \}^{-wot}$$

denote the closure of $\pi(U)$ in the weak operator topology in $B(E)$. Then, since G is abelian,

$$\Gamma = \bigcap_{U \in \mathcal{U}(e)} U_\pi$$

is a set of commuting operators. When endowed with the weak operator topology, it is a compact topological space. Furthermore, if E is given its weak topology, then the action of Γ on E , i. e. the map

$$\begin{aligned} (S, \xi) &\mapsto S\xi, \\ \Gamma \times E &\rightarrow E, \end{aligned}$$

is separately continuous.

Lemma 4.1.1 *E_c coincides with the set of Γ -fixed points in E .*

Proof: If $\xi \in E_c$, then for $\epsilon > 0$ there exists an $U \in \mathcal{U}(e)$ such that $\| \pi(x)\xi - \xi \| < \epsilon$ for all $x \in U$. But then $\| S\xi - \xi \| < \epsilon$ for all $S \in U_\pi$, and it follows that $S\xi = \xi$ for all $s \in \Gamma$.

On the other hand, if $\{x_\alpha\}_{\alpha \in I}$ is a net in G converging to the identity, then the accumulation points of the net of operators $\{\pi(x_\alpha)\}_{\alpha \in I}$, there exists at least one since Γ is compact, are in Γ .

We claim that for any $\xi \in E$ which is fixed by all elements of Γ we have

$$\xi = \lim_{\alpha} \pi(x_\alpha)\xi.$$

Establishing the claim will prove the Lemma, and to do so we are going to prove that all weak topology accumulation points of the net $\{\pi(x_\alpha)\xi\}_{\alpha \in I}$ are contained in

$$\bigcap_{U \in \mathcal{U}(e)} U_\pi \xi = \Gamma \xi = \{\xi\}.$$

For this let $\eta \in E$ be an accumulation point of $\{\pi(x_\alpha)\xi\}_{\alpha \in I}$ in the weak topology of E . Given $U \in \mathcal{U}(e)$, we find, for any weak topology neighbourhood W of η , some α , such that

$$x_\alpha \in U \text{ and } \pi(x_\alpha)\xi \in W.$$

Hence,

$$U_\pi \xi \cap W \neq \emptyset$$

for any such W and we infer, since $U_\pi \xi$ is closed as the continuous image of the compact set U_π , that $\eta \in U_\pi \xi$. \square

For $\xi \in E$ let $\text{conv}\{\pi(U)\xi\}^-$ be the norm closure of the convex hull of $\pi(U)\xi$. This set is bounded, weakly closed because of its convexity, and hence weakly compact in the reflexive space E .

Let

$$C_\xi := \bigcap_{U \in \mathcal{U}(e)} \text{conv}\{\pi(U)\xi\}^-.$$

This then is a non-void, convex, weakly compact and Γ -invariant subset of E reducing to

$$C_\xi = \{\xi\} \text{ if } \xi \in E_c.$$

Furthermore

$$C_{\alpha\xi} = \alpha C_\xi, \quad \alpha \in \mathbb{C}, \xi \in E, \quad (4.1)$$

$$C_{\xi+\eta} \subset C_\xi + C_\eta, \quad \xi, \eta \in E. \quad (4.2)$$

For $\xi \in E$ the Markov-Kakutani fixed-point theorem, see e.g. Chap. IV, Appendix, Theorem 1 of [8], may be applied to the action of Γ on C_ξ . Hence there exists at least one point in C_ξ fixed by all $S \in \Gamma$. This, by Lemma 4.1.1, shows

$$C_\xi \cap E_c \neq \emptyset.$$

We claim that $C_\xi \cap E_c$ contains exactly one element.

If there are $\eta, \zeta \in C_\xi \cap E_c$, then for any $\epsilon > 0$ there exists $U_0 \in \mathcal{U}(e)$ such that for all $x \in U_0$

$$\|\pi(x)\eta - \eta\| < \epsilon \quad \text{and} \quad \|\pi(x)\zeta - \zeta\| < \epsilon.$$

By the definition of C_ξ there are approximations $\tilde{\eta}, \tilde{\zeta} \in \text{conv}\{\pi(U_0)\xi\}$ such that

$$\|\tilde{\eta} - \eta\| < \epsilon \quad \text{and} \quad \|\tilde{\zeta} - \zeta\| < \epsilon.$$

We may write $\tilde{\eta} = \sum_{i=1}^n \lambda_i \pi(y_i)\xi$ and $\tilde{\zeta} = \sum_{j=1}^m \mu_j \pi(z_j)\xi$, where $y_1, \dots, y_n, z_1, \dots, z_m \in U_0$, and $\lambda_1, \dots, \lambda_n > 0$, $\mu_1, \dots, \mu_m > 0$ fulfil $\sum_{i=1}^n \lambda_i = \sum_{j=1}^m \mu_j = 1$.

Then

$$\begin{aligned}
& \left\| \eta - \sum_{j=1}^m \mu_j \pi(z_j) \tilde{\eta} \right\| \\
& \leq \left\| \eta - \sum_{j=1}^m \mu_j \pi(z_j) \eta \right\| + \left\| \sum_{j=1}^m \mu_j \pi(z_j) (\eta - \tilde{\eta}) \right\| \\
& \leq \sum_{j=1}^m \mu_j \sup_j \left\| \eta - \pi(z_j) \eta \right\| + \sum_{j=1}^m \mu_j \sup_j \left\| \pi(z_j) \right\| \left\| \eta - \tilde{\eta} \right\| \\
& \leq \epsilon + C\epsilon.
\end{aligned}$$

Similarly we obtain

$$\left\| \zeta - \sum_{i=1}^n \lambda_i \pi(y_i) \tilde{\zeta} \right\| \leq (1 + C)\epsilon.$$

Since G is commutative

$$\sum_{j=1}^m \mu_j \pi(z_j) \tilde{\eta} = \sum_{j=1}^m \sum_{i=1}^n \mu_j \lambda_i \pi(z_j y_i) \xi = \sum_{i=1}^n \lambda_i \pi(y_i) \tilde{\zeta}$$

coincide and we infer

$$\left\| \eta - \zeta \right\| \leq 2(1 + C)\epsilon,$$

which proves the claim since $\epsilon > 0$ is arbitrary.

Now we have almost proved the following proposition.

Proposition 4.1.1 *Let E be a reflexive Banach space, G a commutative topological group and $\pi : G \rightarrow B(E)$ a uniformly bounded representation of G as above. Then there exists a projection $Q \in B(E)$ with range E_c such that:*

- (i) $\pi(x)Q = Q\pi(x), \quad x \in G,$
- (ii) $\|Q\| \leq \sup_{x \in G} \|\pi(x)\|,$
- (iii) $Q\xi \in C_\xi, \quad \xi \in E.$

Proof: For all $\xi \in E$ we know that $E_c \cap C_\xi$ is a one point set, and hence we may define a map $Q : E \rightarrow E$ by requiring for $\xi \in E$

$$\{Q\xi\} = E_c \cap C_\xi.$$

By its definition it is clear that $Q\xi \in C_\xi$ for all $\xi \in E$ and from the equation (4.1) the homogeneity of Q is obvious.

To see that Q is additive, we claim that for ξ and η in E we have

$$Q\xi + Q\eta \in C_{\xi+\eta},$$

since then we can infer from the uniqueness that

$$Q\xi + Q\eta = Q(\xi + \eta).$$

To establish the claim, it suffices to show that for arbitrary $U \in \mathcal{U}(e)$

$$Q\xi + Q\eta \in \text{conv}\{\pi(U)(\xi + \eta)\}^-.$$

To do this, denote $C = \sup_{x \in G} \|\pi(x)\|$, and choose for $\epsilon > 0$ a symmetric $V \in \mathcal{U}(e)$ such that $V^2 \subset U$ and $\|Q\xi - \pi(x)Q\xi\| < \epsilon$, $\|Q\eta - \pi(x)Q\eta\| < \epsilon$ for all $x \in V$. Now there exist approximations $\tilde{\xi} = \sum_{i=1}^n \lambda_i \pi(y_i)\xi$ and $\tilde{\eta} = \sum_{j=1}^m \mu_j \pi(z_j)\eta$ such that

$$\|Q\xi - \tilde{\xi}\| < \frac{\epsilon}{C} \quad \text{and} \quad \|Q\eta - \tilde{\eta}\| < \frac{\epsilon}{C}.$$

Here again $y_1, \dots, y_n, z_1, \dots, z_m \in V$, and $\lambda_1, \dots, \lambda_n > 0$, $\mu_1, \dots, \mu_m > 0$ fulfil $\sum_{i=1}^n \lambda_i = 1$ and $\sum_{j=1}^m \mu_j = 1$. We note that

$$\sum_{j=1}^m \sum_{i=1}^n \mu_j \lambda_i \pi(z_j y_i)(\xi + \eta) \in \text{conv}\{\pi(V^2)(\xi + \eta)\} \subset \text{conv}\{\pi(U)(\xi + \eta)\},$$

and estimate, using the commutativity of G again:

$$\begin{aligned} & \left\| Q\xi + Q\eta - \sum_{j=1}^m \sum_{i=1}^n \mu_j \lambda_i \pi(z_j y_i)(\xi + \eta) \right\| \\ & \leq \left\| Q\xi - \sum_{j=1}^m \mu_j \pi(z_j) Q\xi \right\| + \left\| \sum_{j=1}^m \mu_j \pi(z_j) \left(Q\xi - \sum_{i=1}^n \lambda_i \pi(y_i) \xi \right) \right\| \\ & \quad + \left\| \sum_{i=1}^n \lambda_i \pi(y_i) \left(Q\eta - \sum_{j=1}^m \mu_j \pi(z_j) \eta \right) \right\| + \left\| Q\eta - \sum_{i=1}^n \lambda_i \pi(y_i) Q\eta \right\| \\ & \leq \epsilon + \sum_{j=1}^m \mu_j C \|Q\xi - \tilde{\xi}\| + \sum_{i=1}^n \lambda_i C \|Q\eta - \tilde{\eta}\| + \epsilon \leq 4\epsilon. \end{aligned}$$

Since $Q^2 = Q$ is evident, it is now proved that Q is a linear projection onto E_c .

That Q is in the commutant of $\pi(G)$ is implied by

$$C_{\pi(x)\xi} = \pi(x)C_\xi, \quad x \in G, \xi \in E$$

which again is a consequence of the commutativity of G .

Finally the norm estimate (ii) is obvious from

$$\sup\{\|\eta\| : \eta \in C_\xi\} \leq \sup_{x \in G} \|\pi(x)\| \|\xi\|, \quad \xi \in E.$$

□

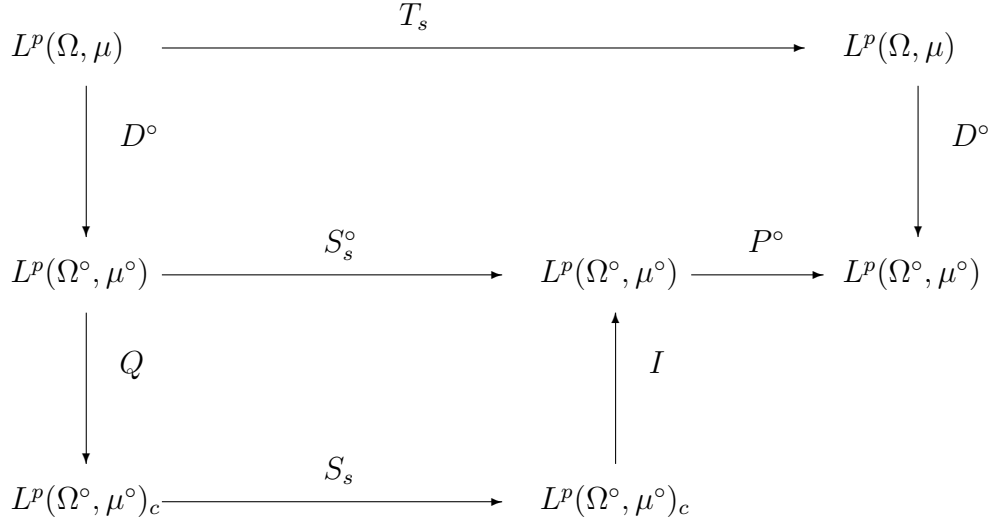


Figure 4.1: The Part of Continuity

4.2 Application to Dilations of continuous one-parameter Semigroups

In this section we shall collect our foregoing considerations to prove the following dilation theorem for continuous one-parameter semigroups:

Theorem 4.2.1 *Let $\{T_t : t \geq 0\}$, with $T_0 = \text{id}$, be a strongly continuous semigroup of positive contractions acting on $L^p(\Omega, \mu)$, where $1 < p < \infty$. Then there exists a measure space $(\tilde{\Omega}, \tilde{\mathfrak{A}}, \tilde{\mu})$ and a strongly continuous group of isometries $\{S_s : s \in \mathbb{R}\}$, with $S_0 = \text{id}$, acting on $L^p(\tilde{\Omega}, \tilde{\mu})$ such that*

$$D \circ T_t = P \circ S_t \circ D, \quad t \geq 0,$$

where D is an isometric embedding of $L^p(\Omega, \mu)$ into $L^p(\tilde{\Omega}, \tilde{\mu})$ and $P : L^p(\tilde{\Omega}, \tilde{\mu}) \rightarrow L^p(\tilde{\Omega}, \tilde{\mu})$ is a contractive projection; further D , P and the isometries $\{S_s : s \in \mathbb{R}\}$ can be chosen to be positive.

Given a semigroup as in the theorem

$$T_t : L^p(\Omega, \mu) \rightarrow L^p(\Omega, \mu), \quad t \geq 0,$$

we obtain for all $s \in \mathbb{Q}_+$ the commutative diagram displayed in Figure 4.1.

Here $\{S_s^\circ : s \in \mathbb{Q}\}$ is the group constructed in Lemma 3.3.1 D° and P° equally come from there, whereas $L^p(\Omega^\circ, \mu^\circ)_c$ and the projection Q are constructed in Proposition 4.1.1. The map I is just the inclusion of the subspace $L^p(\Omega^\circ, \mu^\circ)_c$ and the operators $S_s = S_s^\circ|_{L^p(\Omega^\circ, \mu^\circ)_c}$ are just the restrictions of the $\{S_s^\circ : s \in \mathbb{Q}\}$ to this subspace.

The following lemma shows that the range of D° is included in the set of continuously translating elements of $L^p(\Omega^\circ, \mu^\circ)$.

Lemma 4.2.1 *For $f \in L^p(\Omega, \mu)$, $S^\circ \circ D^\circ(f) : s \mapsto S_s^\circ \circ D^\circ(f)$ is continuous from $\mathbb{Q} \subset \mathbb{R}$ to $L^p(\Omega^\circ, \mu^\circ)$ with its norm topology.*

Proof: Since $\{S^\circ : s \in \mathbb{Q}\}$ is a group of isometries, it suffices to show, for $f \in L^p(\Omega, \mu)$, the continuity from the right of the map $s \mapsto S_s^\circ(D^\circ(f))$ at $s = 0$.

The space $\prod_{\text{LIM}} L^p(\Omega'_{1/n}, \mu'_{1/n}) = L^p(\Omega^\circ, \mu^\circ)$ is uniformly convex. Thus to $\epsilon > 0$ there exists $\eta_p(\epsilon) > 0$ such that for any $f, h \in L^p(\Omega^\circ, \mu^\circ)$ of norm one $\|\frac{1}{2}(f + h)\|_p \geq 1 - \eta_p(\epsilon)$ implies $\|f - h\|_p \leq \epsilon$.

Given a norm one element in the range of D° this may be written $D^\circ(f)$ for an $f \in L^p(\Omega, \mu)$ with $\|f\|_p = 1$. Then for $\epsilon > 0$ there exists $\delta > 0$ such that $\|T_s f - f\|_p \leq 2\eta_p(\epsilon)$ whenever $0 \leq s \leq \delta$. Hence, for $s \in \mathbb{Q} \cap [0, \delta)$,

$$\begin{aligned} \|S_s^\circ(D^\circ(f)) + D^\circ(f)\|_p &\geq \|P^\circ \circ S_s^\circ \circ D^\circ(f) + P^\circ \circ D^\circ(f)\|_p \\ &= \|D^\circ \circ T_s(f) + D^\circ(f)\|_p = \|T_s(f) + f\|_p \\ &= \|2f - (f - T_s(f))\|_p \\ &\geq \|2f\|_p - \|T_s(f) - f\|_p \\ &\geq 2 - 2\eta_p(\epsilon). \end{aligned}$$

Since $\|S_s^\circ(D^\circ(f))\|_p = \|D^\circ(f)\|_p = 1$, we infer $\|S_s^\circ(D^\circ(f)) - D^\circ(f)\|_p \leq \epsilon$. □

Now we may continue with the proof of Theorem 4.2.1.

Proof of Theorem 4.2.1: The above lemma asserts that the range of Q includes $D^\circ(L^p(\Omega, \mu))$. Therefore, denoting $Y := L^p(\Omega^\circ, \mu^\circ)$,

$$Q \circ D^\circ \circ T_s = Q \circ P|_Y^\circ \circ S_s^\circ \circ Q \circ D^\circ, \quad s \in \mathbb{Q}^+.$$

The range of Q is a sub-lattice, as follows from Lemma 6, Chap 6 §17, of Lacey's book [31], and it is closed, since Q is a contractive projection. As an abstract L^p -space it is, by the Bohnenblust-Nakano theorem, see e.g. Chap 5 §15, Theorem 3 of [31], isometrically and order isomorphic to $L^p(\tilde{\Omega}, \tilde{\mu})$, for some measure space $(\tilde{\Omega}, \tilde{\mathfrak{A}}, \tilde{\mu})$, by a linear map Φ , say. Define

$$\begin{aligned} D &= \Phi^{-1} \circ D^\circ, \\ P &= \Phi^{-1} \circ P^\circ|_Y \circ \Phi, \\ S_s &= \Phi^{-1} \circ Q \circ S_s^\circ|_Y \circ \Phi, \quad s \in \mathbb{Q}. \end{aligned}$$

By continuity, the representation S_\cdot can be extended to a continuous representation of \mathbb{R} , still acting on $L^p(\tilde{\Omega}, \tilde{\mu})$. By abuse of notation this extension will still be denoted S_\cdot .

For all $f \in L^p(\Omega, \mu)$ we obtain

$$D \circ T_t(f) = P \circ S_t \circ D(f), \quad t \in \mathbb{R},$$

since both sides are continuous functions of $t \in \mathbb{R}$ and the above equality is valid for the dense subgroup \mathbb{Q} . □

In the last part of this section we indicate the changes necessary, for proving the analogue of Theorem 4.2.1 for a semigroup $\{T_t : t \geq 0\}$ of sub-positive contractions. In this case it can be shown, by the same reasoning as above, that Q is a contractive projection. From (iii) of our proposition it can be seen to be a sub-positive contraction, even. Anyway, in the case $1 < p < \infty$ and $p \neq 2$, the structural description of the range of contractive projections on \mathcal{L}^p -spaces, cf. e.g. chap. 6, §17, Theorem 3 [31], guarantees that, for a direct sum $U : L^p(\Omega^\circ, \mu^\circ) \rightarrow L^p(\Omega^\circ, \mu^\circ)$ of unitary multiplication operators, UY is isometrically and order isomorphic to $L^p(\tilde{\Omega}, \tilde{\mu})$, for some measure space $(\tilde{\Omega}, \tilde{\mathfrak{A}}, \tilde{\mu})$, by an isomorphism which we again call Φ . We note that $U|_Y$ is invertible and acts as the identity on the range of D , since Q does, as follows from Lemma 4.2.1. All we have to do is a further conjugation of $P^\circ|_Y$ and of $\{Q \circ S_s^\circ|_Y : s \in \mathbb{Q}\}$ with $U|_Y$ when defining P and $\{S_s : s \in \mathbb{Q}\}$ before we conjugate the respective results with Φ .

If $p = 2$, then the closed subspace Y is isomorphic to $l^2(I)$ for some set I . In this case no assertion on (sub)positivity properties of the involved operators can be made and we simply transport the group $\{S_s : s \in \mathbb{Q}\}$ by means of this isomorphism.

The completion to a representation of \mathbb{R} and the last conclusion on the strong continuity of this representation can be done exactly as before.

Chapter 5

Transference

5.1 Transference for linear Maps

Our theme is now to obtain reasonable norm estimates for operators which can be defined by strongly convergent integrals

$$\int_0^{\infty} k(t) T_t dt, \quad k \in L^1(\mathbb{R}_+, \lambda).$$

Here $(T_t)_{t \geq 0}$ is still a one-parameter semigroup, strongly continuous, of sub-positive contractions, acting on $E = L^p(\Omega, \mu)$. We shall be merely interested in the case that $1 < p < \infty$, though some results are valid for $p = 1$ too. Often in this case only slight modifications of the proofs are necessary, but for semigroups on \mathcal{L}^1 -spaces we did not prove an analogue of Theorem 4.2.1. Hence we can not take advantage of reducing the problems given for one-parameter semigroups to problems for one-parameter groups.

We display the dilation given by that theorem, respectively its sub-positive version alluded to in the last part of section 4.2 in a commutative diagram (see Figure 5.1). Then it is clear that for all $k \in L^1(\mathbb{R}_+, \lambda)$

$$\left\| \int_0^{\infty} k(t) T_t dt \right\| \leq \left\| \int_0^{\infty} k(t) S_t dt \right\|,$$

and we only need to estimate the latter.

For this let for $k \in L^1(\mathbb{R}, \lambda)$ denote $\lambda_p(k) : L^p(\mathbb{R}, \lambda) \rightarrow L^p(\mathbb{R}, \lambda)$ the convolution with k :

$$\lambda_p(k)f(x) = \int_{\mathbb{R}} k(y)f(x-y) dy, \quad f \in L^p(\mathbb{R}, \lambda), x \in \mathbb{R}.$$

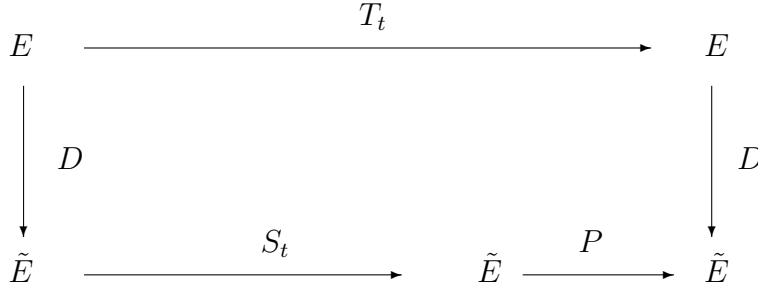


Figure 5.1: Dilation for Reals

Further we denote by

$$\|k\|_{p,p} = \sup \left\{ \|\lambda_p(k)f\|_p : \|f\|_p = 1 \right\}$$

its operator norm on $L^p(\mathbb{R}, \lambda)$.

Theorem 5.1.1 *Assume $1 < p < \infty$ and let $E = L^p(\Omega, \mu)$ be an \mathcal{L}^p -space. If $\pi : \mathbb{R} \rightarrow B(E)$ is a uniformly bounded strongly continuous representation of \mathbb{R} on E , then, with $C = \sup \{\|\pi(x)\| : x \in \mathbb{R}\}$, there holds for all $k \in L^1(\mathbb{R}, \lambda)$:*

$$\left\| \int_{\mathbb{R}} k(x) \pi(x) dx \right\| \leq C^2 \|k\|_{p,p}.$$

Remark 5.1.1 We do not prove the above theorem here since it is a special case of a result which we shall establish (c.f. Theorem 5.2.1) soon. But we would like to remark that for $p = 1$ a stronger estimate

$$\left\| \int_{\mathbb{R}} k(x) \pi(x) dx \right\| \leq C \|k\|_{1,1}$$

holds true for all $k \in L^1(\mathbb{R}, \lambda)$, simply because the norm $\|\cdot\|_{1,1}$ coincides with the L^1 -norm.

The following corollary is due to Coifman and Weiss [14]. Note that the case $p = 1$ holds true for the reasons mentioned in the last remark.

Corollary 5.1.1 (Coifman-Weiss) *Let $(T_t)_{t \geq 0}$ be a strongly continuous semigroup of (sub)positive contractions acting on $E \in \mathcal{L}^p$, where $1 \leq p < \infty$. Then for all $k \in L^1(\mathbb{R}_+, \lambda)$:*

$$\left\| \int_0^\infty k(t) T_t dt \right\| \leq \|k\|_{p,p}.$$

5.2 Complete Boundedness of Transference

For the convenience of the reader we recall the notion of p -complete boundedness from [20] and [43]. To this end we have to introduce some notation.

Assume $1 \leq p < \infty$ and let E be a Banach space, then for $n \in \mathbb{N}$ and an $n \times n$ matrix $(m_{i,j})_{i,j=1}^n \in M_n \otimes B(E)$ of operators $m_{i,j} \in B(E)$ denote

$$\left\| (m_{i,j})_{i,j=1}^n \right\|_{(n)} = \sup \left(\sum_{i=1}^n \left\| \sum_{j=1}^n m_{i,j}(g_j) \right\|^p \right)^{\frac{1}{p}}, \quad (5.1)$$

the supremum being taken over all $g_1, \dots, g_n \in E$, with

$$\left(\sum_{j=1}^n \|g_j\|^p \right)^{\frac{1}{p}} \leq 1.$$

The reader may note that the above norm arises from considering the matrix of operators

$$(m_{i,j})_{i,j=1}^n$$

as acting on the Banach space $l_n^p(E)$. If $E = L^p(\Omega, \mu)$ is a space of class \mathcal{L}^p then $l_n^p(E)$ is clearly again of the same class. Therefore the notion we define next seems to be most useful if the Banach spaces involved are \mathcal{L}^p -spaces or belong to some related class of Banach spaces, e.g. subspaces or subspaces of quotient spaces of \mathcal{L}^p -spaces.

Definition 5.2.1 *Assume that E, F are Banach spaces and that $1 \leq p < \infty$. If $S \subset B(E)$ is a subspace, then we call a linear operator $u : S \rightarrow B(F)$ p -completely bounded if there exists a finite constant $C > 0$ such that for all $n \in \mathbb{N}$ and for all $(m_{i,j})_{i,j=1}^n \in M_n \otimes S$*

$$\left\| (u(m_{i,j}))_{i,j=1}^n \right\|_{(n)} \leq C \left\| (m_{i,j})_{i,j=1}^n \right\|_{(n)}.$$

We denote $\|u\|_p$ the least such constant.

Since the results presented in this section, and their proofs, are almost verbatim the same in the more general situation when one is concerned with an amenable locally compact group and a left Haar measure on it, instead of the locally compact abelian (hence amenable) additive group \mathbb{R} with the (translation invariant) Lebesgue measure, we chose this generality and we let denote G a locally compact amenable group endowed with a Haar measure λ on it.

For $k \in L^1(G, \lambda)$ and $g \in L^p(G, \lambda)$ the convolution product

$$k \star g(x) = \int_G k(y)g(y^{-1}x) d\lambda(y), \quad x \in G$$

is defined. Further, $\lambda_p : k \mapsto (g \mapsto k \star g)$ is injective from $L^1(G, \lambda)$ into $B(L^p(G, \lambda))$ and we may consider its range $\lambda_p(L^1(G, \lambda))$ as a normed subspace of $B(L^p(G, \lambda))$.

A continuous representation of G on $L^p(\Omega, \mu)$ is, by definition, a group homomorphism π mapping G into the invertible operators on $L^p(\Omega, \mu)$ which is continuous when those are endowed with the strong operator topology. If, furthermore, π is uniformly bounded, i.e. $\sup_{x \in G} \|\pi(x)\| < \infty$, then we can consider its extension by integration, $\lambda_\pi : L^1(G, \lambda) \rightarrow B(L^p(\Omega, \mu))$, defined by

$$\lambda_\pi(k)f = \int_G k(x)\pi(x)f d\lambda(x), \quad f \in L^p(\Omega, \mu), \quad k \in L^1(G, \lambda).$$

We remark that λ_π is an algebra homomorphism for the convolution structure on $L^1(G, \lambda)$.

Theorem 5.2.1 *Assume $1 \leq p < \infty$. Let $\pi : G \rightarrow B(L^p(\Omega, \mu))$ be a continuous uniformly bounded representation of G . Then*

$$\lambda_\pi : \lambda_p(L^1(G, \lambda)) \rightarrow B(L^p(\Omega, \mu))$$

is a p -completely bounded algebra homomorphism with norm

$$\|\lambda_\pi\|_p \leq \sup_{x \in G} \|\pi(x)\|^2.$$

Proof: We apply the amenability of the group G in a manner closely related to a Følner-Leptin condition. This seems to be due to C. Herz [28], compare also [14].

For $1 < p < \infty$ let q be defined by $\frac{1}{p} + \frac{1}{q} = 1$. Then for $\alpha \in L^p(G, \lambda)$ and $\beta \in L^q(G, \lambda)$ the convolution $\beta \star \alpha^\vee$, where $\alpha^\vee(x) := \alpha(x^{-1})$, $x \in G$, coincides λ almost everywhere with a continuous function vanishing at infinity, and $\|\beta \star \alpha^\vee\|_\infty \leq \|\alpha\|_p \|\beta\|_q$.

This holds true for $p = 1$, in which case $q = \infty$, given that β has support of finite Haar measure.

Since G is amenable, there exist nets $(\alpha_\tau)_{\tau \in \Delta} \subset L^p(G, \lambda)$ and $(\beta_\tau)_{\tau \in \Delta} \subset L^q(G, \lambda)$ with

$$\sup_{\tau \in \Delta} \|\alpha_\tau\|_p \leq 1, \quad \sup_{\tau \in \Delta} \|\beta_\tau\|_q \leq 1$$

such that

$$\lim_{\tau \in \Delta} \beta_\tau \star \alpha_\tau^\vee = 1$$

uniformly on compact sets.

Hence, whenever $k \in L^1(G, \lambda)$, $f \in L^p(\Omega, \mu)$ and $g \in L^q(\Omega, \mu)$ are given, then there holds true:

$$\begin{aligned} & \int_{\Omega} \lambda_{\pi}(k) f(\omega) g(\omega) \, d\mu(\omega) \\ &= \lim_{\tau \in \Delta} \int_{\Omega} \int_G \beta_{\tau} \star \alpha_{\tau}^{\vee}(x) k(x) \pi(x) f(\omega) g(\omega) \, d\lambda(x) d\mu(\omega) \\ &= \lim_{\tau \in \Delta} \int_{\Omega} \lambda_{\pi}((\beta_{\tau} \star \alpha_{\tau}^{\vee}) \cdot k) f(\omega) g(\omega) \, d\mu(\omega), \end{aligned}$$

where \cdot denotes the pointwise product of functions. This is an abuse of the dominated convergence theorem, since Δ might be uncountable. But here we are concerned with one, later with finitely many, integrable functions on G . They all vanish λ almost everywhere outside a σ -compact subgroup for which one can arrange Δ to be a sequence. Similar arguments justify the use of Fubini's theorem in the arguments given below.

If we denote by $\pi^t : G \rightarrow B(L^q(\Omega, \mu))$ the representation adjoint to π , given by

$$\begin{aligned} & \int_{\Omega} f(\omega) (\pi^t(x) g)(\omega) \, d\mu(\omega) \\ &= \int_{\Omega} (\pi(x^{-1}) f)(\omega) g(\omega) \, d\mu(\omega), \quad x \in G, \end{aligned}$$

then

$$\begin{aligned} & \int_{\Omega} \lambda_{\pi}(\beta \star \alpha^{\vee} \cdot k) f(\omega) g(\omega) \, d\mu(\omega) \\ &= \int_{\Omega} \int_G \int_G k(yx) F^{\omega}(x^{-1}) G^{\omega}(y) \, d\lambda(x) d\lambda(y) d\mu(\omega), \end{aligned}$$

where

$$\begin{aligned} F^{\omega}(x) &= \alpha(x) \pi(x^{-1}) f(\omega), \quad x \in G, \quad \omega \in \Omega \\ G^{\omega}(y) &= \beta(y) \pi^t(y^{-1}) g(\omega), \quad y \in G, \quad \omega \in \Omega. \end{aligned}$$

Finally for $f_1, \dots, f_n \in L^p(\Omega, \mu)$, $g_1, \dots, g_n \in L^q(\Omega, \mu)$ and $(k_{i,j})_{i,j=1}^n \in M_n \otimes \lambda_p(L^1(G, \lambda))$ we compute, with $F_{\tau,j}^{\omega}$ and $G_{\tau,i}^{\omega}$ defined in analogy to the above functions F^{ω} and G^{ω} :

$$\begin{aligned} & \left| \sum_{i,j=1}^n \int_{\Omega} \lambda_{\pi}(k_{i,j})(x) f_j(\omega) g_i(\omega) \, d\mu(\omega) \right| \\ &= \lim_{\tau \in \Delta} \left| \sum_{i,j=1}^n \int_{\Omega} \lambda_{\pi}((\beta_{\tau} \star \alpha_{\tau}^{\vee}) \cdot k_{i,j}) f_j(\omega) g_i(\omega) \, d\mu(\omega) \right| \\ &= \lim_{\tau \in \Delta} \left| \int_{\Omega} \sum_{i,j=1}^n \int_G \int_G k_{i,j}(yx) F_{\tau,j}^{\omega}(x^{-1}) G_{\tau,i}^{\omega}(y) \, d\lambda(x) d\lambda(y) d\mu(\omega) \right|. \end{aligned}$$

We dominate this by

$$\begin{aligned}
& \lim_{\tau \in \Delta} \int_{\Omega} \left\{ \left\| (\lambda_p(k_{i,j}))_{i,j=1}^n \right\|_{(n)} \left(\sum_{j=1}^n \int_G |F_{\tau,j}^{\omega}(x)|^p d\lambda(x) \right)^{\frac{1}{p}} \right. \\
& \quad \left. \left(\sum_{i=1}^n \int_G |G_{\tau,i}^{\omega}(y)|^q d\lambda(y) \right)^{\frac{1}{q}} \right\} d\mu(\omega) \\
& \leq \left\| (\lambda_p(k_{i,j}))_{i,j=1}^n \right\|_{(n)} \lim_{\tau \in \Delta} \left\{ \left(\sum_{j=1}^n \int_G |\alpha_{\tau}(x)|^p \int_{\Omega} |\pi(x^{-1})f_j(\omega)|^p d\mu(\omega) d\lambda(x) \right)^{\frac{1}{p}} \right. \\
& \quad \left. \left(\sum_{i=1}^n \int_G |\beta_{\tau}(y)|^q \int_{\Omega} |\pi^t(y^{-1})g_i(\omega)|^q d\mu(\omega) d\lambda(y) \right)^{\frac{1}{q}} \right\} \\
& \leq \left\| (\lambda_p(k_{i,j}))_{i,j=1}^n \right\|_{(n)} \lim_{\tau \in \Delta} \left(\sup_{x \in G} \|\pi(x^{-1})\| \|\alpha_{\tau}\|_p \left(\sum_{j=1}^n \|f_j\|_p^p \right)^{\frac{1}{p}} \right. \\
& \quad \left. \sup_{y \in G} \|\pi^t(y^{-1})\| \|\beta_{\tau}\|_q \left(\sum_{i=1}^n \|g_i\|_q^q \right)^{\frac{1}{q}} \right) \\
& \leq \sup_{x \in G} \|\pi(x)\|^2 \left\| (\lambda_p(k_{i,j}))_{i,j=1}^n \right\|_{(n)} \left(\sum_{j=1}^n \|f_j\|_p^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n \|g_i\|_q^q \right)^{\frac{1}{q}}.
\end{aligned}$$

For the case $p = 1$, i.e. $q = \infty$, this estimate has to be modified accordingly. \square

Similarly as Theorem 5.1.1 combined with the Dilation Theorem 4.2.1 had a nice conclusion which we stated as Corollary 5.1.1, we here obtain a strengthening of that corollary for p in the range $1 < p < \infty$.

Corollary 5.2.1 *Let $1 < p < \infty$ and $\{T_t : t \geq 0\}$ be a strongly continuous semigroup of positive or subpositive contractions acting on $L^p(X, \mu)$. If for some $n \in \mathbb{N}$ and some matrix $(k_{i,j})_{i,j=1}^n \in M_n \otimes L^1(\mathbb{R}, \lambda)$, whose entries $k_{i,j} \in L^1(\mathbb{R}, \lambda)$ have support in \mathbb{R}_+ , there exists a constant $C \geq 0$ such that for any n elements $g_1, \dots, g_n \in L^p(\mathbb{R}, \lambda)$*

$$\left(\sum_{i=1}^n \int_{\mathbb{R}} \left| \sum_{j=1}^n \lambda_p(k_{i,j}) g_j(y) \right|^p dy \right)^{\frac{1}{p}} \leq C \left(\sum_{j=1}^n \int_{\mathbb{R}} |g_j(y)|^p dy \right)^{\frac{1}{p}},$$

then for all $f_1, \dots, f_n \in L^p(X, \mu)$

$$\left(\sum_{i=1}^n \left\| \sum_{j=1}^n \int_0^{\infty} k_{i,j}(t) T_t f_j dt \right\|_p^p \right)^{\frac{1}{p}} \leq C \left(\sum_{j=1}^n \|f_j\|_p^p \right)^{\frac{1}{p}}.$$

Proof: Let $\{T_t : t \geq 0\}$ be the semigroup under consideration. We have to prove that for a matrix $(k_{i,j})_{i,j=1}^n \in M_n \otimes L^1(\mathbb{R}^+, \lambda)$

$$\left\| \left(\int_0^\infty k_{i,j}(t) T_t dt \right)_{i,j=1}^n \right\|_{(n)} \leq \left\| (\lambda_p(k_{i,j}))_{i,j=1}^n \right\|_{(n)}.$$

Let

$$D \circ T_t = P \circ S_t \circ D, \quad t \geq 0,$$

be a dilation according to Theorem 4.2.1, respectively according to the remarks in the last part of section 4.2. Since $\{T_t : t \geq 0\}$ and $\{S_t : t \in \mathbb{R}\}$ are strongly continuous (semi)groups:

$$\begin{aligned} D \circ \int_0^\infty k(t) T_t dt &= \int_0^\infty k(t) D \circ T_t dt \\ &= \int_0^\infty k(t) D \circ S_t \circ P dt \\ &= D \circ \int_0^\infty k(t) S_t dt \circ P \\ &= D \circ S(k) \circ P, \quad k \in L^1(\mathbb{R}^+, \lambda). \end{aligned}$$

Hence, by Theorem 5.2.1, applied to the continuous representation S of \mathbb{R}

$$\begin{aligned} \left\| \left(\int_0^\infty k_{i,j}(t) T_t dt \right)_{i,j=1}^n \right\|_{(n)} &= \left\| \left(\int_0^\infty D \circ k_{i,j}(t) T_t dt \right)_{i,j=1}^n \right\|_{(n)} \\ &= \left\| (D \circ S(k_{i,j}) \circ P)_{i,j=1}^n \right\|_{(n)} \\ &\leq \left\| (S(k_{i,j}))_{i,j=1}^n \right\|_{(n)} \\ &\leq \left\| (\lambda_p(k_{i,j}))_{i,j=1}^n \right\|_{(n)} \end{aligned}$$

which completes the proof of Corollary 5.2.1. □

5.3 Transference of Square Functions

In this section we shall apply our knowledge about p -completely bounded maps to the transference of certain square functions.

We are more interested to give some examples than to obtain sharp results. In fact, as far as representations of amenable groups or representations of continuous one-parameter semigroups on \mathcal{L}^1 -spaces are concerned our results are not optimal. There are two reasons for this.

- (i) For continuous one-parameter semigroups on \mathcal{L}^1 -spaces we did not prove the existence of a dilation to a strongly continuous one-parameter group. Our constructions heavily involved reflexivity and hence our proof of the subsequently stated result of Coifman and Weiss [14], our Corollary 5.3.1, does not cover the case $p = 1$.
- (ii) Using p -completely bounded maps to derive square function estimates, we pass by inclusions of finite dimensional Hilbert spaces as norm closed subspaces of \mathcal{L}^p -spaces. More even, we use a uniform norm bound on the canonical projection onto these subspaces (c.f. Lemma 5.3.2).

For the case of representations of amenable groups results for p in the whole range $p \in [1, \infty)$, with a better constant than ours, have been obtained by Asmar, Berkson and Gillespie in [6].

Let r_1, r_2, r_3, \dots denote the Rademacher functions, defined by:

$$r_1(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} < t \leq 1, \end{cases}$$

and for $n \geq 1$:

$$r_{n+1}(t) = \begin{cases} r_n(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ r_n(2t-1) & \text{if } \frac{1}{2} < t \leq 1 \end{cases}.$$

The Khinchin inequalities then assert that for $1 \leq p < \infty$ there exist constants c_p and C_p such that for all sequences $(\alpha_i)_{i \in \mathbb{N}} \subset \mathbb{C}$:

$$c_p \left(\int_0^1 \left| \sum_{i=1}^{\infty} \alpha_i r_i(t) \right|^p dt \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{\infty} |\alpha_i|^2 \right)^{\frac{1}{2}} \leq C_p \left(\int_0^1 \left| \sum_{i=1}^{\infty} \alpha_i r_i(t) \right|^p dt \right)^{\frac{1}{p}}.$$

Remark 5.3.1 Trivially $c_p = 1$ for $1 \leq p \leq 2$ and $C_p = 1$ for $2 \leq p < \infty$. The best possible constants have been computed, for $p = 1$ by Szarek [52] and for the other cases by Haagerup [23].

Let $E = L^p(\Omega, \mu)$ be an \mathcal{L}^p -space ($1 < p < \infty$) and for $k = (k_1, \dots, k_n) \in B(E)^n$ let $m^k \in M_{2^n} \otimes B(E)$ be defined by:

$$m_{ij}^k = \begin{cases} \left(\frac{1}{2^n} \right)^{\frac{1}{p}} \sum_{l=1}^n r_l \left(\frac{i}{2^n} \right) k_l & \text{if } j = 1 \\ 0 & \text{if } j \neq 1 \quad \text{for } i, j \in \{1, \dots, 2^n\}. \end{cases} \quad (5.2)$$

So that m is a row-matrix of operators in $M_{2^n} \otimes B(E)$ which space is normed according to (5.1).

Lemma 5.3.1 *With the above notations the following inequalities hold true:*

$$c_p \|m^k\|_{(2^n)} \leq \sup \left\{ \left\| \left(\sum_{j=1}^{2^n} |k_j(g)|^2 \right)^{\frac{1}{2}} \right\|_E : \|g\|_E \leq 1 \right\} \leq C_p \|m^k\|_{(2^n)}.$$

Proof: We take $k = (k_1, \dots, k_n) \in B(E)^n$, form $m \in M_{2^n} \otimes B(E)$ accordingly (we suppress the super index k here and in the following proof), and for $g_1, \dots, g_{2^n} \in E$ we compute:

$$\begin{aligned} c_p^p \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} m_{ij}(g_j) \right\|_E^p &= c_p^p \sum_{i=1}^{2^n} \int_{\Omega} |m_{i1}(g_1)(\omega)|^p d\mu(\omega) \\ &= c_p^p \sum_{i=1}^{2^n} \int_{\Omega} \frac{1}{2^n} \left| \sum_{l=1}^n r_l\left(\frac{i}{2^n}\right) k_l(g_1)(\omega) \right|^p d\mu(\omega) \\ &= c_p^p \int_{\Omega} \int_0^1 \left| \sum_{l=1}^n r_l(t) k_l(g_1)(\omega) \right|^p dt d\mu(\omega) \\ &\leq \int_{\Omega} \left(\sum_{j=1}^n |k_j(g_1)(\omega)|^2 \right)^{\frac{p}{2}} d\mu(\omega) \\ &\leq C_p^p \int_{\Omega} \int_0^1 \left| \sum_{l=1}^n r_l(t) k_l(g_1)(\omega) \right|^p dt d\mu(\omega) \\ &= C_p^p \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} m_{ij}(g_j) \right\|_E^p \end{aligned}$$

Hence on one hand

$$\begin{aligned} c_p \|m\|_{(2^n)} &= c_p \sup \left\{ \left(\sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} m_{ij}(g_j) \right\|_E^p \right)^{\frac{1}{p}} : \sum_{j=1}^{2^n} \|g_j\|_E^p \leq 1 \right\} \\ &\leq \sup \left\{ \left\| \left(\sum_{j=1}^n |k_j(g_1)|^2 \right)^{\frac{1}{2}} \right\|_E : \sum_{j=1}^{2^n} \|g_j\|_E^p \leq 1 \right\} \\ &\leq \sup \left\{ \left\| \left(\sum_{j=1}^n |k_j(g_1)|^2 \right)^{\frac{1}{2}} \right\|_E : \|g_1\|_E \leq 1 \right\}. \end{aligned}$$

And on the other hand, choosing $g_1 = g$, $g_2 = \dots = g_{2^n} = 0$

$$\begin{aligned} \left\| \left(\sum_{j=1}^n |k_j(g)|^2 \right)^{\frac{1}{2}} \right\|_E &\leq C_p \|m\|_{(2^n)} \left(\sum_{j=1}^{2^n} \|g_j\|_E^p \right)^{\frac{1}{p}} \\ &\leq C_p \|m\|_{(2^n)} \|g\|_E. \end{aligned}$$

□

Theorem 5.3.1 *If E, F are \mathcal{L}^p -spaces, $1 \leq p < \infty$, $S \subset B(E)$ is a closed subspace and $u : S \rightarrow B(F)$ a p -completely bounded linear map, then for any sequence $k_1, k_2, \dots \in S$:*

$$\begin{aligned} & \sup \left\{ \left\| \left(\sum_{j=1}^{\infty} |u(k_j)(g)|^2 \right)^{\frac{1}{2}} \right\|_F : \|g\|_F \leq 1 \right\} \\ & \leq \frac{C_p}{c_p} \|u\|_p \sup \left\{ \left\| \left(\sum_{j=1}^{\infty} |k_j(f)|^2 \right)^{\frac{1}{2}} \right\|_E : \|f\|_E \leq 1 \right\}. \end{aligned}$$

Proof: It suffices to prove this for a finite sequence $k_1, \dots, k_n \in S$. For, if $k_1, \dots \in S$ is infinite, then for any $g \in F$ the sequence of functions $(h_n)_{n \in \mathbb{N}}$,

$$h_n = \left(\sum_{j=1}^n |u(k_j)(g)|^2 \right)^{\frac{p}{2}} \in L^1,$$

is non-decreasing and the monotone convergence theorem can be applied. But then, by Lemma 5.3.1, the above left hand side is dominated by

$$\begin{aligned} C_p \left\| (n_{i,j})_{i,j=1}^{2^n} \right\|_{(2^n)} & \leq \|u\|_p C_p \left\| (m_{i,j})_{i,j=1}^{2^n} \right\|_{(2^n)} \\ & \leq \frac{C_p}{c_p} \|u\|_p \sup \left\{ \left\| \left(\sum_{j=1}^{2^n} |k_j(f)|^2 \right)^{\frac{1}{2}} \right\|_E : \|f\|_E \leq 1 \right\}, \end{aligned}$$

where $m = (m_{i,j})_{i,j=1}^{2^n}$ is constructed from the sequence $k_1, \dots, k_n \in S$ and $n = (n_{i,j})_{i,j=1}^{2^n} = (u(m_{i,j}))_{i,j=1}^{2^n} = \text{id} \otimes u \left((m_{i,j})_{i,j=1}^{2^n} \right)$ from $u(k_1), \dots, u(k_n) \in B(F)$ just as in (5.2) before the lemma. \square

As mentioned in the introduction to this section we are not able to derive from the above theorem and from Corollary 5.2.1 the case $p = 1$ of a theorem of Coifman, Weiss (their Corollary 4.17. in [14]). By our means we only can prove:

Corollary 5.3.1 *Let $(T_t)_{t \geq 0}$ be a strongly continuous semigroup of positive or sub-positive contractions on $E = L^p(\Omega, \mu)$, where $1 < p < \infty$. If $k_1, k_2, \dots \in L^1(\mathbb{R}, \lambda)$ with $\text{supp } k_i \subset [0, \infty)$, $i \in \mathbb{N}$ are such that for some $M > 0$ for all $f \in L^p(\mathbb{R}, \lambda)$*

$$\int_{\mathbb{R}} \left(\sum_{i=1}^{\infty} |\lambda_p(k_i)f(x)|^2 \right)^{\frac{p}{2}} dx \leq M^p \int_{\mathbb{R}} |f(x)|^p dx,$$

then for all $f \in L^p(\Omega, \mu)$ we have

$$\left\| \left(\sum_{i=1}^{\infty} \left| \int_0^{\infty} k_i(t) T_t f dt \right|^2 \right)^{\frac{1}{2}} \right\|_E \leq M \frac{C_p}{c_p} \|f\|_E.$$

Proof: The Corollary 5.2.1 of the last section just states that the map

$$u : \lambda_p(k) \mapsto \int_0^\infty k(t)T_t dt,$$

defined on $S = \{\lambda_p(k) : \text{supp}(k) \in [0, \infty), k \in L^1(\mathbb{R}, \lambda)\} \subset B(L^p(\mathbb{R}, \lambda))$, is p-completely bounded from S to $B(E)$. Hence, by the above theorem, the assertion of the corollary is immediate. \square

Remark 5.3.2

- (i) The corollary has an obvious counterpart for uniformly bounded strongly continuous representations of amenable locally compact groups $\pi : G \rightarrow B(E)$ on \mathcal{L}^p -spaces E . Here the p-complete boundedness of $\pi : \lambda_p(L^1(G)) \rightarrow B(E)$ follows directly from Theorem 5.2.1. We leave it to the reader to formulate and prove this (the case $p = 1$ can be included). For a different, more direct approach to this we refer the reader to the paper [6] of Asmar, Berkson and Gillespie.
- (ii) In the proof of Lemma 5.3.1 we used just the embeddings of l_n^2 in $L^p([0, 1])$ given by the Rademacher functions. If, instead of Rademacher functions, appropriate Gaussian random variables are used, then one can avoid the appearance of the constant $\frac{C_p}{c_p}$ in the final result. (This requires an additional approximation procedure, since a finite subset of the considered random variables should be realizable on a finite set).

Similar to the former we may consider the problem of transferring, by means of a p-completely bounded $u : S \rightarrow B(L^p(\Omega', \mu'))$, square-function inequalities of the following type:

Given a sequence $k_1, k_2, \dots \in S \subset B(L^p(\Omega, \mu))$ such that for some M

$$\left\| \left(\sum_{i=1}^\infty |k_i(f_i)|^2 \right)^{\frac{1}{2}} \right\|_p \leq M \left\| \left(\sum_{i=1}^\infty |f_i|^2 \right)^{\frac{1}{2}} \right\|_p$$

for all sequences $f_1, f_2, \dots \in L^p(\Omega, \mu)$, we may ask whether a similar inequality holds true for the sequence $u(k_1), u(k_2), \dots \in B(L^p(\Omega', \mu'))$, may be with some additional constant depending on p only.

Whereas in our last problem we discussed, for varying $n \in \mathbb{N}$ and $k = (k_1, \dots, k_n)$, operators

$$\begin{aligned} K_k : L^p(\Omega, \mu) &\rightarrow L^p([0, 1], \lambda; L^p(\Omega, \mu)), \\ K_k : f &\mapsto \sum_{i=1}^n r_i(\cdot) k_i(f), \end{aligned}$$

we shall now consider

$$\begin{aligned} K'_k : L^p([0, 1], \lambda; L^p(\Omega, \mu)) &\rightarrow L^p([0, 1], \lambda; L^p(\Omega, \mu)), \\ K'_k : F &\mapsto \int_0^1 \sum_{i=1}^n r_i(\cdot) k_i(F(t)) r_i(t) dt. \end{aligned}$$

Thus, for a sequence $k = (k_i)_{i \in \mathbb{N}} \subset B(L^p(\Omega, \mu))$ denote $\|k\|_{[2]}$ the least constant M such that for all $f_1, f_2, \dots \in L^p(\Omega, \mu)$:

$$\left\| \left(\sum_{j=1}^{\infty} |k_j(f_j)|^2 \right)^{\frac{1}{2}} \right\|_p \leq M \left\| \left(\sum_{j=1}^{\infty} |f_j|^2 \right)^{\frac{1}{2}} \right\|_p.$$

That is, $\|k\|_{[2]}$ is the norm of the sequence k as an operator on $L^p(\Omega, \mu; l^2)$.

Proposition 5.3.1 *There exist constants d_p and D_p depending only on p , $1 < p < \infty$, such that*

$$d_p \|K'_k\| \leq \|k\|_{[2]} \leq D_p \|K'_k\|$$

for all sequences $k = (k_i)_{i \in \mathbb{N}} \subset B(L^p(\Omega, \mu))$.

To prove the proposition we note first that for the finite sequence $l = (\text{id}, \dots, \text{id})$ the operator $K'_l = P_n \otimes \text{id}$ is just the tensor product of the projection $P_n : L^p([0, 1], \lambda) \rightarrow L^p([0, 1], \lambda)$ onto the span of the first n Rademacher functions and of the identity on $L^p(\Omega, \mu)$. Thus, for $F \in L^p([0, 1], \lambda; L^p(\Omega, \mu))$,

$$K'_l(F)(s) = P_n \otimes \text{id}(F)(s) = \sum_{i=1}^n \int_0^1 F(t) r_i(t) dt r_i(s), \quad s \in [0, 1],$$

and the Khinchin inequalities imply:

Lemma 5.3.2 *If $p \in (1, \infty)$, then for all $F \in L^p([0, 1], \lambda; L^p(\Omega, \mu))$:*

$$\|P_n \otimes \text{id}(F)\| \leq \max\{c_p^{-1}, c_q^{-1}\} \|F\|,$$

where $q \in (1, \infty)$ is such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: If $2 \leq p$, then for μ almost all $\omega \in \Omega$

$$\begin{aligned} c_p^p \int_0^1 |P_n \otimes \text{id}(F)(t, \omega)|^p dt &\leq \left(\int_0^1 |P_n \otimes \text{id}(F)(t, \omega)|^2 dt \right)^{\frac{p}{2}} \\ &\leq \left(\int_0^1 |F(t, \omega)|^2 dt \right)^{\frac{p}{2}} \\ &\leq \int_0^1 |F(t, \omega)|^p dt, \end{aligned}$$

since P_n is an orthogonal projection on the Hilbert space $L^2([0, 1], \lambda)$. Integrating this with respect to μ proves the lemma in this case.

For p in the range $1 < p < 2$ the assertion is then established by duality: Let $q \in [2, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ then for $G \in L^q([0, 1], \lambda; L^q(\Omega, \mu))$:

$$\begin{aligned} \left| \int_{\Omega} \int_0^1 P_n \otimes \text{id}(F)(t, \omega) G(t, \omega) dt d\omega \right| &= \left| \int_{\Omega} \int_0^1 F(t, \omega) P_n \otimes \text{id}(G)(t, \omega) dt d\omega \right| \\ &\leq \|F\|_p \|P_n \otimes \text{id}(G)\|_q \\ &\leq c_q^{-1} \|F\|_p \|G\|_q. \end{aligned}$$

By the converse to Hölder's inequality we obtain the assertion. □

Proof of Proposition 5.3.1: As in the proof of Theorem 5.3.1 we may assume that $k = (k_1, \dots, k_n) \subset B(E)$ is finite. Let $f_1, \dots, f_n \in E = L^p(\Omega, \mu)$ be given. Then, by the orthogonality relations of the Rademacher functions,

$$\begin{aligned} K'_k \left(\sum_{j=1}^n r_j f_j \right) (s, \omega) &= \int_0^1 \sum_{i=1}^n \sum_{j=1}^n r_i(s) k_i(f_j)(\omega) r_i(t) r_j(t) dt \\ &= \sum_{i=1}^n r_i(s) k_i(f_i(\omega)). \end{aligned}$$

Hence we obtain the right hand side inequality of the proposition, with $D_p = \frac{C_p}{c_p}$, from

$$\begin{aligned} \int_{\Omega} \left(\sum_{i=1}^n |k_i(f_i)(\omega)|^2 \right)^{\frac{p}{2}} d\mu(\omega) &\leq C_p^p \int_{\Omega} \int_0^1 \left| \sum_{i=1}^n r_i(t) k_i(f_i)(\omega) \right|^p dt d\mu(\omega) \\ &\leq C_p^p \left\| K'_k \left(\sum_{j=1}^n r_j f_j \right) \right\|_{L^p([0,1], \lambda; E)}^p \\ &\leq C_p^p \|K'_k\|^p \left\| \left(\sum_{j=1}^n r_j f_j \right) \right\|_{L^p([0,1], \lambda; E)}^p \\ &\leq \left(\frac{C_p}{c_p} \right)^p \|K'_k\|^p \left\| \sum_{i=1}^n (|f_i|^2)^{\frac{1}{2}} \right\|_E^p. \end{aligned}$$

To deduce the other estimate we apply the Lemma 5.3.2. Given $F \in L^p([0, 1], \lambda; E)$ define $f_i = \int_0^1 F(t) r_i(t) dt \in L^p(\Omega, \mu)$, so that

$$P_n \otimes \text{id}(F)(s, \omega) = \sum_{i=1}^n r_i(s) \otimes f_i(\omega).$$

Then,

$$\begin{aligned}
\|K'_k F\|_{L^p([0,1],\lambda;E)}^p &= \|K'_k P_n \otimes \text{id}(F)\|_{L^p([0,1],\lambda;E)}^p \\
&= \int_{\Omega} \int_0^1 \left| \sum_{i=1}^n r_i(s) k_i(f_i)(\omega) \right|^p ds d\mu(\omega) \\
&\leq c_p^{-p} \int_{\Omega} \left(\sum_{i=1}^n |k_i(f_i)(\omega)|^2 \right)^{\frac{p}{2}} d\mu(\omega) \\
&\leq c_p^{-p} \|k\|_{[2]}^p \int_{\Omega} \left(\sum_{i=1}^n |f_i(\omega)|^2 \right)^{\frac{p}{2}} d\mu(\omega) \\
&\leq c_p^{-p} \|k\|_{[2]}^p \int_{\Omega} C_p^p \int_0^1 \left(\left| \sum_{i=1}^n r_i(t) f_i(t, \omega) \right|^p \right) dt d\mu(\omega) \\
&= \left(\frac{C_p}{c_p} \right)^p \|k\|_{[2]}^p \|P_n \otimes \text{id}(F)\|_{L^p([0,1],\lambda;E)}^p \\
&\leq \left(\frac{C_p}{c_p} \right)^p \max\{c_p^{-1}, c_q^{-1}\}^p \|k\|_{[2]}^p \|F\|_{L^p([0,1],\lambda;E)}^p.
\end{aligned}$$

Where now $d_p = \left(\frac{c_p}{C_p} \right) \min\{c_p, c_q\}$. □

The analogue of Theorem 5.3.1 is:

Theorem 5.3.2 *For $1 < p < \infty$ there exists a constant C_p^* such that, whenever $E, F \in \mathcal{L}^p$ are \mathcal{L}^p -spaces, $S \subset B(E)$ is a closed subspace, and $u : S \rightarrow B(F)$ is a p -completely bounded linear map, then for all sequences $g_1, g_2, \dots \in F$:*

$$\left\| \left(\sum_{j=1}^{\infty} |u(k_j)(g_j)|^2 \right)^{\frac{1}{2}} \right\|_F \leq C_p^* M \left\| \left(\sum_{j=1}^{\infty} |g_j|^2 \right)^{\frac{1}{2}} \right\|_F,$$

whenever $k_1, k_2, \dots \in S$ is a sequence for which there exists M such that for all $f_1, f_2, \dots \in E$:

$$\left\| \left(\sum_{j=1}^{\infty} |k_j(f_j)|^2 \right)^{\frac{1}{2}} \right\|_E \leq M \left\| \left(\sum_{j=1}^{\infty} |f_j|^2 \right)^{\frac{1}{2}} \right\|_E.$$

Proof: It suffices again to consider finite sequences

$$k = (k_1, k_2, \dots, k_n) \subset S^n.$$

Then K'_k may be viewed as an element of $M_{2^n} \otimes S$, and denoting $u(k) = (u(k_1), \dots, u(k_n))$ we have

$$\text{id}_{2^n} \otimes u(K'_k) = K'_{u(k)}.$$

In fact, the sigma algebra Σ_n generated by the first n Rademacher functions contains just 2^n atoms of Lebesgue measure 2^{-n} . If we denote \mathcal{E}_n the conditional expectation $\mathcal{E}_n : L^p([0, 1], \mathcal{B}, \lambda) \rightarrow L^p([0, 1], \Sigma_n, \lambda)$, then

$$(\mathcal{E}_n \otimes \text{id}) \circ K'_k \circ (\mathcal{E}_n \otimes \text{id}) = K'_k \text{ for all } k = (k_1, \dots, k_n) \in B(E)^n.$$

Denote $\delta_1, \dots, \delta_{2^n}$ the standard unit vector basis of $l_{2^n}^p$ and $I_n : l_{2^n}^p \rightarrow L^p([0, 1], \Sigma_n, \lambda)$ the canonical identification

$$I_n : \delta_i \mapsto 2^{\frac{n}{p}} \chi_{(\frac{i-1}{2^n}, \frac{i}{2^n}]}, \quad i = 1, \dots, 2^n.$$

Then $(I_n \otimes \text{id})^{-1} \circ K'_k \circ (I_n \otimes \text{id})$ is given by a matrix of operators

$$m^k = (m_{i,j}^k)_{i,j=1}^{2^n} \in M_{2^n} \otimes B(E),$$

where

$$m_{ij}^k = \sum_{l=1}^{2^n} r_l\left(\frac{i}{2^n}\right) r_l\left(\frac{j}{2^n}\right) k_l \quad \text{for } i, j \in \{1, \dots, 2^n\}. \quad (5.3)$$

Thus, by the p -complete boundedness of u :

$$\|m^{u(k)}\| \leq \|u\|_p \|m^k\|.$$

Since

$$\|K'_k\| = \|(\mathcal{E}_n \otimes \text{id}) \circ K'_k \circ (\mathcal{E}_n \otimes \text{id})\| = \|m^k\|,$$

an application of Proposition 5.3.1 yields:

$$\begin{aligned} \|u(k)\|_{[2]} &\leq D_p \|K'_{u(k)}\| \leq D_p \|u\|_p \|K'_k\| \\ &\leq \frac{D_p}{d_p} \|k\|_{[2]} \leq \frac{D_p}{d_p} M. \end{aligned}$$

This proves the theorem with the constant $C_p^* = \frac{D_p}{d_p}$. □

Remark 5.3.3 For the convenience of the reader we formulate in this remark a result due to Asmar, Berkson and Gillespie [6] (item (i) below). That theorem holds true for $p \in [1, \infty)$. But for reasons discussed in the introduction to this section, it appears as a corollary, with an additional constant, to Theorem 5.3.2 only for $1 < p < \infty$.

- (i) Let G be a locally compact, amenable group and $\pi : G \rightarrow B(E)$ a strongly continuous uniformly bounded representation of G on a space $E \in \mathcal{L}^p$.

If, for a sequence $k_1, k_2, \dots \in L^1(G, \lambda)$, there exists M such that for all $f_1, f_2, \dots \in L^p(G, \lambda)$:

$$\int_G \left(\sum_{i=1}^{\infty} |\lambda_p(k_i) f_i(x)|^2 \right)^{\frac{p}{2}} d\lambda(x) \leq M \int_G \left(\sum_{i=1}^{\infty} |f_i(x)|^2 \right)^{\frac{p}{2}} d\lambda(x),$$

then for all $g_1, g_2, \dots \in E$:

$$\left\| \left(\sum_{i=1}^{\infty} |\pi(k_i)g_i|^2 \right)^{\frac{1}{2}} \right\|_E \leq M \sup_{g \in G} \|\pi(g)\|^2 \left\| \left(\sum_{i=1}^{\infty} |g_i|^2 \right)^{\frac{1}{2}} \right\|_E.$$

- (ii) This time we leave it to the reader to formulate the version of the last theorem for one-parameter semigroups.

5.4 Transference of Maximal Functions

The story of studying dilation theorems for positive contractions on \mathcal{L}^p -spaces had been initiated by Akçoglu [1] as a means to prove a maximal ergodic theorem:

Theorem 5.4.1 (Akçoglu) *Let $1 \leq p < \infty$ (the case $p = \infty$ is trivial), $E = L^p(\Omega, \mu) \in \mathcal{L}^p$ and let $T : E \rightarrow E$ be a positive contraction.*

Then, for some constant c_p depending only on p , there holds true for all $f \in L^p(\Omega, \mu)$:

$$\left[\int_{\Omega} \left(\sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^n |T^k f(\omega)| \right)^p d\mu(\omega) \right]^{\frac{1}{p}} \leq c_p \left(\int_{\Omega} |f(\omega)|^p d\mu(\omega) \right)^{\frac{1}{p}}.$$

Remark 5.4.1

- (i) If T is a contraction for $p = 1$ and for $p = \infty$, then the assertion of this theorem essentially is contained in the Hopf-Dunford-Schwartz maximal ergodic theorem. The above theorem then follows simply by interpolation. Its value lies in the fact that it deals with a single p .
- (ii) If T is an invertible isometry, then the conclusion of the theorem had been shown to hold true by A. Ionescu-Tulcea. Furthermore, Akçoglu reduces, involving a dilation, to this case.
- (iii) A. de la Torre in [53] has a nice proof of the ergodic theorem relying on ideas of transference. This is further developed in the monograph [13] of Coifman, Rochberg and Weiss.
- (iv) Asmar, Berkson and Gillespie in their paper [7] generalise from the group of integers to uniformly bounded representations of amenable groups by separation preserving operators.

We shall consider in this section a locally compact group G and a strongly continuous, uniformly bounded representation

$$\pi : G \rightarrow B(E) \quad (5.4)$$

of it on $E = L^p(\Omega, \mu)$, ($1 \leq p < \infty$), by separation preserving operators. Such a representation we shall call a separation preserving representation.

Alternatively, we could deal in the remainder of this chapter with a representation π of G on E such that

$$\sup_{x \in G} \|\pi(x)\| \leq \infty.$$

We shall not do so, but we shall shortly discuss the definition and properties of the map $x \mapsto |\pi(x)|$, when π is a separation preserving representation.

Definition 5.4.1 *We shall say that a positive operator P dominates another operator R (resp. R is dominated by P), if for all $f \in L^p(\Omega, \mu)$*

$$|Rf| \leq P|f|.$$

Proposition 5.4.1 *If $R : E \rightarrow E$ is separation preserving, then there exists a positive operator P dominating R , with $\|P\| = \|R\|$. Furthermore, $|Rf| = P|f|$ for all $f \in E$, and this justifies to denote this operator $|R|$.*

Proof: The assertion we know already if R is a contraction, see Lemma 3.2.1 and Remark 3.2.4. This we may apply to $\frac{1}{\|R\|} \cdot R$. □

Remark 5.4.2 Now the map $|\pi(\cdot)| : G \rightarrow B(E)$ is defined properly and we should note that it is in fact strongly continuous.

Actually, for a non-negative $f \in L^p(\Omega, \mu)$ and $x, y \in G$ we may estimate μ -almost everywhere:

$$\begin{aligned} ||\pi(x)|f| - |\pi(y)|f| &= ||\pi(x)f| - |\pi(y)f|| \\ &\leq |\pi(x)f - \pi(y)f|. \end{aligned}$$

Hence we obtain the continuity of $x \mapsto |\pi(x)|f$, for non-negative f , from the strong continuity of the representation in question. Since any element of $L^p(\Omega, \mu)$ is a linear combination of at most four non-negative ones, we are done.

For a finite sequence $k_1, \dots, k_l \in L^1(G, \lambda)$ consider the maximal functions

$$m(f)(x) = \sup_{1 \leq i \leq l} |\lambda_p(k_i)f|(x), \quad f \in L^p(G, \lambda), \quad (5.5)$$

$$M(h)(\omega) = \sup_{1 \leq i \leq l} |\pi_p(k_i)h|(x), \quad h \in L^p(\Omega, \pi). \quad (5.6)$$

Our aim is to give an estimate of M in terms of m .

Lemma 5.4.1 *Given the above conditions and non-negative $\alpha \in L^p(G, \lambda)$, $\beta \in L^q(G, \lambda)$, there holds for all $f \in E$ and $i = 1, \dots, l$, μ -almost everywhere:*

$$|\pi((\beta \star \alpha^\vee) \cdot k_i) f| \leq \int_G \beta(y) |\pi(y)| \sup_{1 \leq i \leq l} \left\{ \left| \int_G \alpha^\vee(x) k_i(yx) \pi(x) f d\lambda(x) \right| \right\} d\lambda(y).$$

Here q is such that $\frac{1}{p} + \frac{1}{q} = 1$, and if $q = \infty$, then we additionally assume that $\beta \in L^\infty(G, \lambda)$ has a support of finite Haar measure.

Proof:

$$\begin{aligned} \pi((\beta \star \alpha^\vee) \cdot k_i) f &= \int_G \int_G \beta(y) \alpha^\vee(y^{-1}x) k_i(x) \pi(x) f d\lambda(y) d\lambda(x) \\ &= \int_G \beta(y) \int_G \alpha^\vee(x) k_i(yx) \pi(y) \pi(x) f d\lambda(x) d\lambda(y) \\ &= \int_G \beta(y) \pi(y) \int_G k_i(yx) \alpha^\vee(x) \pi(x) f d\lambda(x) d\lambda(y). \end{aligned}$$

Hence, it follows that

$$\begin{aligned} |\pi((\beta \star \alpha^\vee) \cdot k_i) f| &\leq \int_G \beta(y) |\pi(y)| \left| \int_G k_i(yx) \alpha^\vee(x) \pi(x) f d\lambda(x) \right| d\lambda(y) \\ &\leq \int_G \beta(y) |\pi(y)| \sup_{1 \leq i \leq l} \left| \int_G k_i(yx) \alpha^\vee(x) \pi(x) f d\lambda(x) \right| d\lambda(y), \end{aligned}$$

where the last inequality is true since $|\pi(y)|$ is a positive operator. \square

Theorem 5.4.2 *Assume that G is amenable and that $\pi : G \rightarrow B(E)$ is a representation of G on $E \in \mathcal{L}^p$ as in (5.4). Let $k_1, \dots \in L^1(G, \lambda)$ be a sequence and define the maximal operators corresponding to this sequence as in (5.5) and (5.6). If m_k is a constant such that for all $f \in L^p(G, \lambda)$:*

$$\|m(f)\|_{L^p(G, \lambda)} \leq m_k \|f\|_{L^p(G, \lambda)},$$

then for all $h \in E$:

$$\|M(h)\|_{L^p(\Omega, \mu)} \leq m_k \sup_{y \in G} \|\pi(y)\|^2 \|h\|_{L^p(\Omega, \mu)}.$$

Proof: Approximating $Mh = \lim_{l \rightarrow \infty} M_l h$ from below by functions

$$M_l h(\omega) = \sup_{1 \leq i \leq l} |\pi(k_i) h(\omega)| \quad l = 1, 2, \dots,$$

we see, that it is sufficient to prove the theorem in the case that the sequence $k = (k_1, \dots, k_l)$ is finite.

Now, since G is amenable, there exist sequences $(\alpha_n)_{n \in \mathbb{N}}$, $(\beta_n)_{n \in \mathbb{N}}$, with $0 \leq \alpha_n \in L^p(G, \lambda)$, $0 \leq \beta_n \in L^q(G, \lambda)$, $\|\alpha_n\|_p = \|\beta_n\|_q = 1$ for all $n \in \mathbb{N}$, such that as $n \rightarrow \infty$:

$$\|(\beta_n \star \alpha_n^\vee) \cdot k_i - k_i\|_1 \rightarrow 0 \quad \text{for } i \in \{1, \dots, l\}.$$

(See the proof of Theorem 5.2.1 for this.)

The strong continuity and the uniform boundedness of the representation imply the convergence of $\pi((\beta_n \star \alpha_n^\vee) \cdot k_i)h$ to $\pi(k_i)h$ in L^p -norm for each $i \in \{1, \dots, l\}$. Since the lattice operations are continuous, we infer the L^p -norm convergence of $\sup_{1 \leq i \leq l} |\pi((\beta_n \star \alpha_n^\vee) \cdot k_i)h|$ to $\sup_{1 \leq i \leq l} |\pi(k_i)h|$.

Since $\|M_l h\|_p = \sup \left\{ \int_G g(\omega) M_l h(\omega) d\mu(\omega) : g \geq 0, \|g\|_q = 1 \right\}$, it suffices to estimate (with obvious modifications if $q = \infty$):

$$\int_\Omega g(\omega) \sup_{1 \leq i \leq l} |\pi((\beta_n \star \alpha_n^\vee) \cdot k_i)h|(\omega) d\mu(\omega) \leq m_k \|g\|_q \|h\|_p \sup_{x \in G} \|\pi(x)\|^2.$$

Now, by the lemma:

$$\begin{aligned} & \int_\Omega g(\omega) \sup_{1 \leq i \leq l} |\pi((\beta_n \star \alpha_n^\vee) \cdot k_i)h|(\omega) d\mu(\omega) \\ & \leq \int_\Omega g(\omega) \int_G \beta_n(y) |\pi(y)| \sup_{1 \leq i \leq l} \left| \int_G k_i(yx) \alpha_n^\vee(x) \pi(x) h(\omega) d\lambda(x) \right| d\lambda(y) d\mu(\omega) \\ & = \int_G \beta_n(y) \int_\Omega |\pi(y)|^* g(\omega) \sup_{1 \leq i \leq l} \left| \int_G k_i(yx) \alpha_n^\vee(x) \pi(x) h(\omega) d\lambda(x) \right| d\mu(\omega) d\lambda(y) \\ & \leq \int_G \beta_n(y) \|\pi(y)\|^* \|g\|_q \left\| \sup_{1 \leq i \leq l} \left| \int_G k_i(yx) \alpha_n^\vee(x) \pi(x) h(\omega) d\lambda(x) \right| \right\|_p d\lambda(y) \\ & \leq \left(\int_G |\beta_n(y)|^q \sup_{z \in G} \|\pi(z)\|^* \|g\|_q^q d\lambda(y) \right)^{\frac{1}{q}} \\ & \quad \left(\int_G \int_\Omega \sup_{1 \leq i \leq l} \left| \int_G k_i(yx) \alpha_n^\vee(x) \pi(x) h(\omega) d\lambda(x) \right|^p d\mu(\omega) d\lambda(y) \right)^{\frac{1}{p}} \\ & \leq \sup_{z \in G} \|\pi(z)\|^* \|g\|_q \left(\int_\Omega \|m(\alpha_n(\cdot) \pi(\cdot) h(\omega))\|_p^p d\mu(\omega) \right)^{\frac{1}{p}} \\ & \leq \sup_{z \in G} \|\pi(z)\|^* \|g\|_q \left(\int_\Omega m_k^p \int_G |\alpha_n(x) \pi(x) h(\omega)|^p d\lambda(x) d\mu(\omega) \right)^{\frac{1}{p}} \\ & \leq m_k \sup_{z \in G} \|\pi(z)\|^* \sup_{z \in G} \|\pi(z)\| \|g\|_q \|h\|_p, \end{aligned}$$

and Proposition 5.4.1 implies the statement. □

As a consequence we can prove a version of the Hopf-Dunford-Schwartz maximal ergodic theorem. We shall deal only with the case that our \mathcal{L}^p -spaces are

reflexive, i.e. $p > 1$. For $p = 1$, a probability measure space (Ω, μ) , and a single operator T with $T(1) = 1$, the theorem is due to Chacon and Ornstein [12]. But before stating and proving it we remind the reader of the following simple fact which we shall need in our proof of the theorem.

Lemma 5.4.2 *If $E = L^p(\Omega, \mu)$, $E' = L^p(\Omega', \mu')$ are \mathcal{L}^p -spaces, $1 < p < \infty$, and $D : E \rightarrow E'$ is a positive isometry, then for all real valued $f, g \in E$:*

$$D(f \vee g) = D(f) \vee D(g).$$

Proof: Since D is positive we obtain the inequality

$$D(f \vee g) \geq D(f) \vee D(g) \quad (5.7)$$

from $D(f \vee g) - D(f) = D(f \vee g - f) \geq 0$ and $D(f \vee g) - D(g) = D(f \vee g - g) \geq 0$. Now, let

$$\begin{aligned} A &= \{ \omega : f(\omega) = g(\omega) \}, \\ B &= \{ \omega : f(\omega) \vee g(\omega) = f(\omega) \} \setminus A, \\ C &= \{ \omega : f(\omega) \vee g(\omega) = g(\omega) \} \setminus A. \end{aligned}$$

Clearly,

$$\begin{aligned} D(f \vee g) &= D(f \cdot \chi_A) + D(f \cdot \chi_B) + D(g \cdot \chi_C) \\ &= D(g \cdot \chi_A) + D(f \cdot \chi_B) + D(g \cdot \chi_C). \end{aligned}$$

Since D is positive and isometric we know from Remark 3.2.3 that the supports

$$\begin{aligned} A_1 &= \text{supp } D(f \cdot \chi_A) = \text{supp } D(g \cdot \chi_A), \\ B_1 &= \text{supp } D(f \cdot \chi_B), \\ C_1 &= \text{supp } D(g \cdot \chi_C) \end{aligned}$$

are mutually disjoint. Furthermore,

$$\begin{aligned} A_1 \cap \text{supp } D(f \cdot \chi_{\Omega \setminus A}) &= \emptyset, \\ B_1 \cap \text{supp } D(f \cdot \chi_{\Omega \setminus B}) &= \emptyset, \\ C_1 \cap \text{supp } D(g \cdot \chi_{\Omega \setminus C}) &= \emptyset. \end{aligned}$$

Hence on A_1, B_1, C_1 , respectively, we have:

$$\begin{aligned} D(f \vee g) &= D(f \cdot \chi_A) = D(f) \leq D(f) \vee D(g), \\ D(f \vee g) &= D(f \cdot \chi_B) = D(f) \leq D(f) \vee D(g), \\ D(f \vee g) &= D(g \cdot \chi_C) = D(g) \leq D(f) \vee D(g). \end{aligned}$$

□

Theorem 5.4.3 *Let $(T_t)_{t \geq 0}$ be a strongly continuous one-parameter semigroup of positive contractions acting on some \mathcal{L}^p -space $E = L^p(\Omega, \mu)$. For $p > 1$ there exists a constant C_p , depending only on p , such that for all $f \in E$:*

$$\left\| \sup_{t>0} \left| \frac{1}{t} \int_0^t T_s f \, ds \right| \right\|_E \leq C_p \|f\|_E.$$

Proof: First we note that we are taking a supremum, pointwise, over an uncountable set of functions $\omega \mapsto \frac{1}{t} \int_0^t T_s f(\omega) \, ds$. But $t \mapsto T_t f$ is continuous from $(0, \infty)$ to $L^p(\Omega, \mu)$ and \mathbb{Q}_+ is a countable dense subset in $(0, \infty)$, hence, as Dunford and Schwartz show on page 686 of their work [18]:

$$\sup_{t>0} \left| \frac{1}{t} \int_0^t T_s f \, ds \right| = \sup_{t \in \mathbb{Q}_+} \left| \frac{1}{t} \int_0^t T_s f \, ds \right|, \quad (5.8)$$

except on a possibly f -dependent set of μ -measure 0. We note further, that it suffices to prove that for each finite subset $\{t_1, \dots, t_n\} =: N \subset \mathbb{Q}$:

$$\left\| \sup_{t \in N} \left| \frac{1}{t} \int_0^t T_s f \, ds \right| \right\|_E \leq C_p \|f\|_E.$$

Because then the monotone convergence theorem will provide the estimate for the right hand side of the above equality (5.8).

Now, for $t > 0$, we let $k_t = \frac{1}{t} \chi_{[0,t]}$. The corresponding maximal operator on $L^p(\mathbb{R}, \lambda)$:

$$m(h)(x) = \sup_{t>0} |\lambda_p(k_t)h(x)| = \sup_{t>0} \left| \frac{1}{t} \int_0^t h(x-s) \, ds \right|,$$

is just the left sided Hardy-Littlewood maximal operator, for whose L^p boundedness we refer the reader to [26] and the theorems 384 and 398 of [27].

Now we apply the dilation theorem, Theorem 4.2.1 to the semigroup to write

$$D \circ T_t = P \circ S_t \circ D, \quad t \geq 0.$$

Our preparing Lemma 5.4.2 and Theorem 5.4.2 yield:

$$\begin{aligned}
\left\| \sup_{t \in N} \left\| \frac{1}{t} \int_0^t T_s f \, ds \right\| \right\|_E &= \left\| D \left(\sup_{t \in N} \left\| \frac{1}{t} \int_0^t T_s f \, ds \right\| \right) \right\|_{\tilde{E}} \\
&= \left\| \sup_{t \in N} \left\| \frac{1}{t} \int_0^t DT_s f \, ds \right\| \right\|_{\tilde{E}} \\
&= \left\| \sup_{t \in N} \left\| \frac{1}{t} \int_0^t PS_s Df \, ds \right\| \right\|_{\tilde{E}} \\
&\leq \left\| \sup_{t \in N} \left\| \frac{1}{t} \int_0^t S_s Df \, ds \right\| \right\|_{\tilde{E}} \\
&= \left\| \sup_{t \in N} M(Df) \right\|_{\tilde{E}} \leq C_p \|Df\|_{\tilde{E}} \\
&= C_p \|f\|_E,
\end{aligned}$$

where M is the maximal operator corresponding to the representation $t \mapsto S_t$ and the finite sequence $k = (k_t)_{t \in N}$. □

Chapter 6

Submarkovian Semigroups

6.1 Some Examples of Sub-Markovian Semigroups

For a σ -finite measure space (Ω, μ) we consider a semigroup $(T_t)_{t \geq 0}$ acting “simultaneously” on all $L^p(\Omega, \mu)$ spaces, $1 \leq p \leq \infty$, such that for all $f \in L^2(\Omega, \mu) \cap L^p(\Omega, \mu)$ and all $t \geq 0$:

$$\begin{aligned} i) \quad & \|T_t f\|_p \leq \|f\|_p \\ ii) \quad & T_t^* = T_t \text{ on } L^2(\Omega, \mu) \\ iii) \quad & T_0 = \text{id} \\ iv) \quad & T_t f \geq 0 \text{ if } f \geq 0. \end{aligned} \tag{6.1}$$

Furthermore, for $p < \infty$ we assume strong continuity of the map

$$\begin{aligned} T & : t \mapsto T_t, \\ T & : [0, \infty) \rightarrow L^p(\Omega, \mu). \end{aligned}$$

In this chapter we require σ -finiteness of the measure space to use the duality $L^1(\Omega, \mu)^* = L^\infty(\Omega, \mu)$.

The above conditions are surely to a large extend redundant and to an even larger extend not really needed for our development. To be more precise we define:

Definition 6.1.1 *A one-parameter semigroup $(T_t)_{t \geq 0}$ acting strongly continuously on $L^2(\Omega, \mu)$ is called a submarkovian semigroup if:*

$$\begin{aligned} i) \quad & \|T_t f\|_2 \leq \|f\|_2 \\ ii) \quad & T_t^* = T_t \text{ on } L^2(\Omega, \mu) \\ iii) \quad & T_0 = \text{id} \\ iv) \quad & T_t f \geq 0 \text{ if } f \geq 0 \\ v) \quad & \|T_t f\|_\infty \leq \|f\|_\infty \text{ if } f \in L^2(\Omega, \mu) \cap L^\infty(\Omega, \mu). \end{aligned}$$

Remark 6.1.1

- (i) A use of the selfadjointness, the norm bounds and an application of the Marcinkiewicz interpolation theorem (see e.g. [47] Chap. 5 sect. 2) implies that our original set of conditions is fulfilled by a submarkovian semigroup. Strictly speaking we should notationally distinguish the different extensions $(T_t^p)_{t \geq 0}$, obtained this way on the different L^p -spaces, $1 \leq p < \infty$. But in the sequel it is always understood that on L^p we are considering the extension by continuity of an operator defined on $L^p \cap L^2$.
- (ii) Thus far our presented results are all proved for a single $p \in (1, \infty)$, but now we need a little more to define functions of the negative of the infinitesimal generator $-A_p$ on L^p . At the moment we see at least two “philosophies”:

1. Extend everything from $L^p \cap L^2$.
2. Assume, for one $p \in (1, \infty)$, some analyticity of the semigroup.

In the first, more traditional case, to which we essentially shall stick on, we use von Neumann’s spectral calculus as a start. Then a good set of conditions is the above i)-iii), and instead of iv) and v), that for some $r_0 \in (1, 2)$ $(T_t)_{t \geq 0}$ extends, by continuity, to a semigroup of sub-positive contractions on L^p , $r_0 \leq p \leq r'_0$. We shall return to this in Remark (6.3.1).

In the second case, one can define functions of the generator, in fact more elementarily, by the integral calculus given for the resolvent of the infinitesimal generator. If one still considers only a single $p \in (1, \infty)$, it would be necessary for us to have a semigroup of sub-positive contractions which is strongly continuous on $L^p(\Omega, \mu)$. Some of our following conclusions are improved by interpolation, and such as the maximal theorem, Theorem 6.3.3, essentially need selfadjointness on L^2 .

We denote A the infinitesimal generator of $(T_t)_{t \geq 0}$ on L^2 , such that

$$\begin{aligned} T_t &= e^{tA}, \\ Af &= \lim_{t \searrow 0} \frac{T_t - 1}{t} f, \\ D(A) &= \{f \in L^2(\Omega, \mu) : \lim_{t \searrow 0} \frac{T_t - 1}{t} f \text{ exists in norm}\}. \end{aligned}$$

Then A is a selfadjoint operator with $\sigma(A) \subset \mathbb{R}_- \cup \{0\}$, hence $-A$ is positive.

Further, we know from von Neumann’s spectral theory that there exists a unique spectral resolution of the identity $(P_\lambda)_{\lambda \in \mathbb{R}}$, such that

$$\begin{aligned} Af &= \int_{-\infty}^0 \lambda dP_\lambda f & f \in D(A), \\ T_t f &= \int_{-\infty}^0 e^{t\lambda} dP_\lambda f & f \in L^2. \end{aligned}$$

We shall suppose $P_0 = 0$, i.e. A is $1 - 1$ on $D(A)$.

Next we recall some examples of semigroups, contractive on some \mathcal{L}^p -spaces.

Example 6.1.1

- (i) On $L^p(\mathbb{R}, \lambda)$ let $T_s f(\cdot) = f(\cdot - s)$, $f \in L^p(\mathbb{R}, \lambda)$, $s \geq 0$, denote the semigroup of translation-operators, which are not selfadjoint. Note that we artificially made a one-parameter semigroup out of an, on $L^2(\mathbb{R}, \lambda)$, unitary group.

The generator A is an extension of $-\frac{d}{dx} : f \mapsto -f'$ from the space $\{h : h' \text{ exists and } h' \in L^2(\mathbb{R}, \lambda)\}$. Using the notation $\hat{\cdot} : f \mapsto \hat{f}$ for the Fourier transform, defined on $L^1(\mathbb{R}, \lambda) \cap L^2(\mathbb{R}, \lambda)$ by:

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx, \quad \xi \in \mathbb{R},$$

we have

$$\begin{aligned} (T_t f)^\wedge(\xi) &= e^{-it\xi} f(\xi), \\ (Af)^\wedge(\xi) &= -i\xi \hat{f}(\xi). \end{aligned}$$

- (ii) Convolution with Gaussian kernels on $L^p(\mathbb{R}^n, \lambda)$,

$$T_t f(x) = \frac{1}{(2\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{(x-y)^2}{2t}} f(y) dy.$$

Here the generator A is one half of the Laplacian,

$$Af(x) = \frac{1}{2} \Delta f(x) = \frac{1}{2} \sum_{i=1}^n \left(\frac{\partial}{\partial x_i} \right)^2 f(x).$$

On the side of the Fourier transform:

$$\begin{aligned} (T_t f)^\wedge(\xi) &= e^{-\frac{t}{2}|\xi|^2} \hat{f}(\xi), \\ (Af)^\wedge(\xi) &= -\frac{|\xi|^2}{2} \hat{f}(\xi). \end{aligned}$$

- (iii) Poisson semigroup on $L^p(\mathbb{R}^n, \lambda)$,

$$\begin{aligned} T_t f(x) &= \left(\frac{1}{\pi} \right)^n \int_{\mathbb{R}^n} \frac{t}{(t^2 + |y|^2)^{\frac{n+1}{2}}} f(x - y) dy, \\ A &= -(-\Delta)^{\frac{1}{2}}. \end{aligned}$$

For the Fourier transforms this means:

$$\begin{aligned} (T_t f)^\wedge(\xi) &= e^{-t|\xi|} \hat{f}(\xi), \\ (Af)^\wedge(\xi) &= -|\xi| \hat{f}(\xi). \end{aligned}$$

- (iv) An example not fulfilling all of the above conditions:

On $L^p(\mathbb{R}, \lambda)$ let for $t \geq 0$:

$$S_t f(x) = e^{\frac{t}{p}} f(e^t x).$$

Then, only on the fixed \mathcal{L}^p -space, $\|S_t\| \leq 1$. On that space the semi-group is strongly continuous, and additionally all the S_t are positive operators.

For $f \in C^1$:

$$\begin{aligned} Af(x) &= \lim_{t \searrow 0} \frac{e^{\frac{t}{p}} f(e^t x) - f(x)}{t} \\ &= \lim_{t \searrow 0} \frac{e^{\frac{t}{p}} (f(e^t x) - f(x))}{t} + \frac{e^{\frac{t}{p}} - 1}{t} f(x) \\ &= f'(x) \cdot x + \frac{1}{p} f(x). \end{aligned}$$

- (v) Evolution semigroups to the Schrödinger operator with certain perturbations, for example point interactions are not always bounded on the whole range $1 \leq p < \infty$. In three dimensions one has to deal with a semigroup which is uniformly bounded on $L^p(\mathbb{R}, \lambda)$ only for $p \in (\frac{1}{3}, 3)$. (see [11] and [3] for more on this.)

6.2 Fourier L^p -Multipliers

In this section we shall discuss some properties of distributions which map, by convolution, the space $L^p(\mathbb{R}, \lambda)$ into itself. We shall be interested in the case $1 < p < \infty$. Here, and in the sequel, by a distribution we mean a functional ϕ defined on the Schwartz space $\mathcal{S}(\mathbb{R})$. For information on this class, the class of tempered distributions, we refer the reader to Chap.1 sect.3 of the book of Stein and Weiss [47]. In a lot of cases the distributions, we have to deal with, will be given by integration against a function which we shall denote $x \mapsto \phi(x)$, $x \in \mathbb{R}$.

Definition 6.2.1 *A tempered distribution ϕ is called an L^p -convolver, respectively its Fourier transform $\hat{\phi}$ is called an L^p -multiplier, if for some $C > 0$ for all $f \in \mathcal{S}(\mathbb{R})$*

$$\left\| (\hat{\phi} \cdot \hat{f})^\sim \right\|_p \leq C \|f\|_p.$$

Here, $\sim: \varphi \mapsto \check{\varphi}$ denotes the inverse Fourier transform.

Remark 6.2.1 Sometimes it is useful to note that \mathcal{L}^p -convolvers are just those bounded operators on $L^p(\mathbb{R})$ which commute with all translations (c.f. [47] chap. I).

Some remarks on the space $Cv_p(\mathbb{R})$ of all L^p -convolvers seem to be useful for our development.

Remark 6.2.2

- (i) From our definition it is natural to consider $Cv_p(\mathbb{R})$ as a sub-algebra of the bounded operators on $L^p(\mathbb{R}, \lambda)$. In there, it is not only norm closed, but it can be shown to be closed in the weak operator topology. For $p = 2$ we just deal with the von Neumann algebra of \mathbb{R} .
- (ii) If $(u_k)_{k=1}^\infty$ is a norm bounded approximate identity in $L^1(\mathbb{R}, \lambda)$, and if $\phi \in Cv_p(\mathbb{R})$, then $\phi \star u_k$ tends to ϕ in the strong operator topology, as $k \rightarrow \infty$. This provides a method to approximate a convolver, some awkward distribution, by quite regular ones, i.e. some given by integration against C^∞ -functions.
- (iii) One should be a bit careful when defining the pointwise product of a convolver and a function. That is, it is not always true that this product is again a convolver, or even exists as a distribution, if the function is only assumed to be, say, bounded and continuous.

There is anyway a nice class of functions for which the pointwise product can be defined, moreover, this class constitutes an algebra $B_p(\mathbb{R})$, with respect to pointwise multiplication. This is the so called algebra of Herz-Schur multipliers (see [20], [28], [10] for more on this).

- (iv) A purely operator theoretic definition of $B_p(\mathbb{R})$ can be given as follows: $B_p(\mathbb{R})$ is just the space of weakly continuous, p -completely bounded, pointwise multipliers of $Cv_p(\mathbb{R})$. Though this sounds a bit awkward, simply because of its length, it has been on the roots of introducing the notions of complete and p -complete boundedness. We mention here only that p -completely bounded maps have nice functorial properties.

An alternative characterisation runs as follows: A function φ belongs to $B_p(\mathbb{R})$, if there exists a subspace of a quotient-space of an \mathcal{L}^p -space, call it E , and a strongly continuous, uniformly bounded representation $S : \mathbb{R} \rightarrow B(E)$ on it, such that $\varphi(t) = (S_t \xi, \eta)$, $t \in \mathbb{R}$, for some $\xi \in E$, $\eta \in E^*$ [10] [20].

- (v) It is useful to include “subspaces of quotients” of \mathcal{L}^p -spaces in the above, since any Hilbert space is of this type and hence any continuous positive definite function φ is in $B_p(\mathbb{R})$. Moreover, for some such φ one has: $\|\varphi \cdot \phi\|_{p,p} \leq \varphi(0) \|\phi\|_{p,p}$ for all L^p -convolvers ϕ .

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This gives a tool for a further regularisation of an convolver. Namely it can be boundedly weakly approximated by convolvers with compact support, at least on the amenable group \mathbb{R} .

We shall use the functions $v_j(x) = \max\{1 - \frac{|x|}{j}, 0\}$, $x \in \mathbb{R}$, $j \in \mathbb{N}$ for this.

There are some celebrated theorems providing non-trivial L^p -convolvers. Because of its relation to analytic functions the multiplier theorem of Hörmander, see sect. 2 of [29], will be of importance to us and we want to introduce the reader to this theme next.

Definition 6.2.2 *We say that a distribution ϕ fulfils the Mihlin-Hörmander conditions with constant $C > 0$, if*

$$\begin{aligned} |\hat{\phi}(\nu)| &\leq C, \\ \left| \nu \frac{\partial}{\partial \nu} \hat{\phi}(\nu) \right| &\leq C, \quad \forall \nu \in \mathbb{R}. \end{aligned}$$

The Hörmander multiplier theorem then asserts that, for $1 < p < \infty$,

$$\|\phi \star f\|_p \leq C \cdot C_1[p + (p-1)^{-1}] \|f\|_p \quad \forall f \in L^p.$$

Remark 6.2.3 In the above we wrote on the right hand side the dependence on p of a constant. This dependence enters in the proof of a result of Cowling, which we shall state as Corollary 6.3.1.

The theorem of Hörmander, we in the above referred to, is stated as theorem 2.5 in [29]. Regarding the assumptions of that theorem it would be more correct to call the condition required in there “Hörmander’s condition” and the one we stated in the definition “Mihlin’s condition” (c.f. [38], resp. Theorem 2 [39], Appendix). In our one-dimensional setting the square-integrability condition, Hörmander requires, on the derivative of ϕ is slightly weaker than the uniform growth, respectively decrease, estimate which is required above. It should be noted, however, that in the n -dimensional case Hörmander reduced the differentiability requirement from the order n to the least integer not less than half of the dimension.

For a proof of the multiplier theorem, we used, we refer to Theorem 3 and its corollary in chapter IV §3 of the book of Stein [46].

We agree, if the interested reader finds it hard to chase for the constant there. So we shall give some more arguments here. The multipliers we just consider are, in fact bounded from L^1 to weak- L^1 . Interpolating now with the L^2 – L^2 estimate by means of the Marcinkiewicz interpolation theorem exhibits the dependence of the constant on p . Alternatively one can show that the multipliers are bounded from the real Hardy space $H_{\mathbb{R}}^1$ to L^1 and interpolate again.

If, for $\theta \in (0, \pi)$, Γ_θ denotes the cone

$$\Gamma_\theta = \{ z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta \} = \{ \rho e^{i\psi} : \rho > 0, |\psi| < \theta \},$$

where we represented $z \neq 0$ as $z = \rho e^{i \arg z}$, with $\arg z \in [-\pi, \pi)$, $\rho > 0$, then a bounded holomorphic function m on Γ_θ has almost everywhere non-tangential boundary values and we denote m again the boundary function $m : \partial\Gamma_\theta \rightarrow \mathbb{C}$.

The cone Γ_ψ is strictly contained in Γ_θ for $\psi \in (0, \theta)$, and we define a function m_ψ on \mathbb{R} :

$$m_\psi(x) = \begin{cases} m(xe^{i\psi}) & \text{if } x \geq 0 \\ m(|x|e^{-i\psi}) & \text{if } x < 0. \end{cases} \quad (6.2)$$

Definition 6.2.3 Given m on Γ_θ as above, we define a distribution ϕ by:

$$\phi(h) = \lim_{\psi \nearrow \theta} \int m_\psi(x) \hat{h}(-x) dx, \quad h \in \mathcal{S}.$$

We then shall say that the Fourier transform of ϕ coincides with the boundary values of m on Γ_θ .

Example 6.2.1 An interesting example is the distribution ϕ with Fourier transform

$$\hat{\phi}(\nu) = (i\nu)^{i\gamma}.$$

In this case

$$\phi(x) = \begin{cases} \Gamma(-i\gamma)^{-1} x^{i\gamma-1} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

$\hat{\phi}$ is the boundary value of the, in $\mathbb{C} \setminus [-\infty, 0]$ holomorphic, function

$$m_\gamma(z) = z^{i\gamma} = \exp(i\gamma \log |z| - \gamma \arg z), \quad z \in \mathbb{C}.$$

We note that m fulfils the Mihlin-Hörmander conditions 6.2.2:

$$\begin{aligned} |m_\gamma(i\nu)| &\leq e^{|\gamma|\frac{\pi}{2}}, \\ \left| \frac{\partial}{\partial \nu} m_\gamma(i\nu) \right| &= \left| i \frac{\gamma}{\nu} e^{\pm \gamma \frac{\pi}{2}} \right| \leq \frac{1}{|\nu|} |\gamma| e^{|\gamma|\frac{\pi}{2}} \quad \nu \in \mathbb{R}. \end{aligned}$$

If $\overline{\Gamma_\psi} \subset \Gamma_\theta$ are two cones with $\psi < \theta$, like in Figure 6.1, then the “boundary” value on Γ_ψ of a function which is bounded and holomorphic on the larger cone Γ_θ , fulfils the conditions of Hörmander’s Fourier multiplier theorem. This is a consequence of the Cauchy integral theorem.

In fact, if $\zeta = |x| e^{i\psi} \in \partial\Gamma_\psi$, then a circle C_r with centre ζ and radius r lies in Γ_θ , provided $r < \sin(\theta - \psi) |x|$ (compare Figure 6.1). It follows from (6.3)

and (6.4) that

$$\begin{aligned} |m_\psi(x)| &\leq \frac{1}{2\pi} \int_{C_r} |m(z)| \frac{|dz|}{r} \leq M_\theta, \\ \left| x \frac{d}{dx} m_\psi(x) \right| &\leq \frac{|x|}{2\pi} \int_{C_r} |m(z)| \frac{|dz|}{r^2} \leq M_\theta \frac{|x|}{r}, \end{aligned}$$

where $M_\theta = \sup_{z \in \Gamma_\theta} |m(z)|$ denotes the least upper bound of $|m|$ on Γ_θ . The conditions are thus fulfilled for m_ψ , with bound $M_\theta \sin^{-1}(\theta - \psi)$.

6.3 Application to Functional Calculus

Now let $(T_t)_{t \geq 0}$ be a semigroup, related to a measure space (Ω, μ) , as in section (6.1) which fulfils the conditions i)–iii) of (6.1). We denote A its infinitesimal generator on $L^2(\Omega, \mu)$. If m is a bounded Borel function on $[0, \infty)$, then $\int_{-\infty}^0 m(-\lambda) dP_\lambda = m(-A)$ is defined on $L^2(\Omega, \mu)$. For those semigroups Cowling [15] discusses the question whether $m(-A)$ extends to a bounded operator on $L^p(\Omega, \mu)$ and considers the consequences of this functional calculus. I shall sketch some of his results and proofs in this section.

We first note that no assumption on positivity, as in (6.1) iv) is made. The considered class is thus more general than the class of submarkovian semigroups. But there is still a relation to positivity. Namely, a contraction on L^1 is automatically a sub-positive contraction (see e.g. Theorem 1.1 in chapter 4 §[30]). This enables one to make use of the dilation theorems and their implications.

If $P_0 \neq 0$ then to define $m(-A)$ it would be necessary to know $m(0)$ which might not be defined by the above argumentation. Not to worry about this was the reason to assume $P_0 = 0$. Furthermore, if a uniformly bounded sequence

Cauchy's formulae:

$$\begin{aligned} m(\zeta) &= \frac{1}{2\pi i} \oint_{C_r} \frac{m(z)}{z - \zeta} dz \quad (6.3) \\ \frac{d}{d\zeta} m(\zeta) &= \frac{1}{2\pi i} \oint_{C_r} \frac{m(z)}{(z - \zeta)^2} dz \quad (6.4) \end{aligned}$$

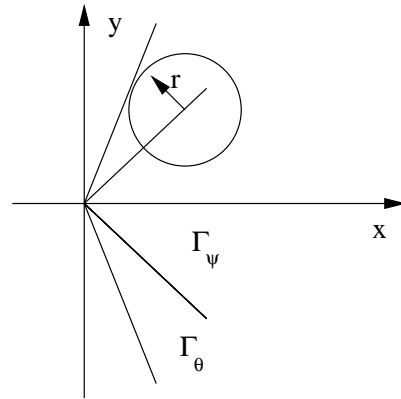


Figure 6.1: Two Cones

$(m_k)_{k \in \mathbb{N}}$ of bounded Borel functions on $(0, \infty)$ converges to some function m uniformly on compact sets in $(0, \infty)$, then $m_k(-A)$ converges to $m(-A)$ in the weak operator topology of $B(L^2(\Omega, \mu))$, since the spectral measure is concentrated on $(-\infty, 0)$.

Now as a corollary of our previous work we obtain a theorem of M. Cowling, Theorem 1 in [15], which is his starting point for discussing the work [46] of E. Stein. Our proof is a variation of that one given by Cowling.

Theorem 6.3.1 *Let m be bounded, holomorphic in $\Gamma_{\frac{\pi}{2}}$. Suppose that the distribution ϕ whose Fourier transform coincides almost everywhere with the boundary values of m fulfils, for some finite constant C_p depending only on p , $1 < p < \infty$,*

$$\|\phi \star f\|_p \leq C_p \|f\|_p \quad \forall f \in C_{cp}^\infty(\mathbb{R}).$$

Then for all $f \in L^p(\Omega, \mu) \cap L^2(\Omega, \mu)$

$$\|m(-A)f\|_p \leq C_p \|f\|_p.$$

Proof: The distribution $\hat{\phi}$ can be extended to a function, still denoted $\hat{\phi}$, which is holomorphic and bounded in the lower half-plane, since m is such in $\Gamma_{\frac{\pi}{2}}$. By the Paley-Wiener theorem the support of ϕ is contained in $[0, \infty)$. Next one regularises ϕ :

(i) If, for $k \in \mathbb{N}$, $u_k \in C_{cp}^\infty$ is such that

$$\text{supp } u_k \in [\frac{1}{k}, \frac{2}{k}] \text{ and } \int_{\mathbb{R}} u_k(x) dx = 1,$$

then

$$\text{supp } \phi \star u_k \subset [\frac{1}{k}, \infty), \phi \star u_k \in L^p \cap C^\infty.$$

Furthermore, for $f \in L^p(\mathbb{R}, \lambda)$ we have convergence in L^p -norm:
 $u_k \star f \rightarrow f$ as $k \rightarrow \infty$.

(ii) If, for $j \in \mathbb{N}$, the function v_j is defined by

$$v_j(x) = \max\{1 - \frac{|x|}{j}, 0\}, \quad x \in \mathbb{R},$$

then each v_j is a positive definite function, and $v_j \nearrow 1$ uniformly on compacts as $j \rightarrow \infty$.

It follows, see Remark 6.2.2, that for all $j, k \in \mathbb{N}$

$$\|v_j(\phi \star u_k)\|_{p,p} \leq \|\phi \star u_k\|_{p,p} \leq \|\phi\|_{p,p} \leq C_p. \quad (6.5)$$

By the conditions imposed in (i) the supports of u_k and $\phi \star u_k$ are contained in the interval $[\frac{1}{k}, \infty) \subset [0, \infty)$. Thus, a bounded holomorphic extension to the lower half-plane \mathbb{C}_- exists for each of them. We note that, for those extensions and for $z \in \mathbb{C}_-$,

$$\begin{aligned} (\phi \star u_k)^\wedge(z) &= \int_0^\infty \phi \star u_k(x) e^{-izx} dx \\ &= \hat{\phi}(z) \hat{u}_k(z). \end{aligned}$$

Hence,

$$\begin{aligned} \int_{-\infty}^\infty (\phi \star u_k)(t) e^{-t\lambda} dt &= \int_{-\infty}^\infty (\phi \star u_k)(t) e^{-it(-i\lambda)} dt \\ &= (\phi \star u_k)^\wedge(-i\lambda) = \hat{\phi}(-i\lambda) \hat{u}_k(-i\lambda) \\ &= m(\lambda) \hat{u}_k(-i\lambda). \end{aligned}$$

Using again the conditions imposed on u_k we see from

$$\hat{u}_k(-i\lambda) = \int_{-\infty}^\infty u_k(t) e^{-it(-i\lambda)} dt = \int_{-\infty}^\infty u_k(t) e^{-t\lambda} dt,$$

that $\hat{u}_k(-i\lambda) \rightarrow 1$ boundedly and uniformly on finite intervals in \mathbb{R}_+ . Now it is not hard to see that

$$\mathcal{S}_{j,k}f := \int_0^\infty (v_j \cdot (\phi \star u_k))(t) T_t f dt$$

converges to $m(-A)f$ in L^2 when first $j \rightarrow \infty$ and then $k \rightarrow \infty$. In fact,

$$\begin{aligned} \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \mathcal{S}_{j,k}f &= \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \int_0^\infty v_j(t) (\phi \star u_k)(t) T_t f dt \\ &= \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \int_0^\infty v_j(\phi \star u_k)(t) \int_{-\infty}^0 e^{t\lambda} dP_\lambda f dt \\ &= \lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \int_{-\infty}^0 e^{t\lambda} \int_0^\infty v_j(\phi \star u_k)(t) dt dP_\lambda f \\ &= \lim_{k \rightarrow \infty} \int_{-\infty}^0 \int_0^\infty e^{t\lambda} (\phi \star u_k)(t) dt dP_\lambda f \\ &= \int_{-\infty}^0 \int_0^\infty e^{t\lambda} \phi(t) dt dP_\lambda f \\ &= \int_{-\infty}^0 m(-\lambda) dP_\lambda f. \end{aligned}$$

For all $j, k \in \mathbb{N}$ we know, using the transference theorem of Coifman and Weiss, our Corollary 5.1.1, that (6.5) implies

$$\|\mathcal{S}_{j,k}f\|_p \leq C_p \|f\|_p.$$

Because of this boundedness we obtain the existence of $\lim_{k \rightarrow \infty} \lim_{j \rightarrow \infty} \mathcal{S}_{j,k}$ in the strong operator topology on the closure of $L^p(\Omega, \mu) \cap L^2(\Omega, \mu)$, that is on all of $L^p(\Omega, \mu)$. The uniqueness of the limit on $L^p(\Omega, \mu) \cap L^2(\Omega, \mu)$ shows that the limit must be the extension to $L^p(\Omega, \mu)$ of the restriction to $L^p(\Omega, \mu) \cap L^2(\Omega, \mu)$ of the spectrally defined $m(-A)$. \square

In the case that m is given as in Example 6.2.1 it follows that

$$(-A)^{i\gamma} = m_\gamma(-A)$$

extends to a bounded operator on $L^q(\Omega, \mu)$, $1 < q < \infty$, of norm at most

$$\|(-A)^{i\gamma}\|_{q,q} \leq C_1[q + (q-1)^{-1}] (1 + \gamma)e^{\gamma \frac{\pi}{2}}. \quad (6.6)$$

Since

$$\|(-A)^{i\gamma}\|_{2,2} = \sup_{x \in \mathbb{R}} |m_\gamma(x)| = 1, \quad (6.7)$$

it is possible to improve this by interpolation.

Corollary 6.3.1 *For some constant $C(p)$, depending only on p , $1 < p < \infty$:*

$$\|(-A)^{i\gamma}\|_{p,p} \leq C(p)[1 + |\gamma|^{12}]^{\frac{1}{p} - \frac{1}{2}} e^{\pi|\frac{1}{p} - \frac{1}{2}||\gamma|}.$$

Proof: We may assume $p \neq 2$.

The function

$$f : y \mapsto \frac{y}{y - \log y}, \quad y \in [e, \infty)$$

decreases monotonically to 1. Thus there exists at most one $\gamma_0 > e$ such that

$$f(\gamma_0) \left| \frac{2}{p} - 1 \right| = 1.$$

If a solution to this equation exists, then we define γ_p equal to it. Otherwise, i.e. if p is such that $f(y) \left| \frac{2}{p} - 1 \right| < 1$ for all $y \geq e$, then we let $\gamma_p = e$.

If $|\gamma| \leq \gamma_p$, then we estimate

$$\begin{aligned} \|(-A)^{i\gamma}\|_{p,p} &\leq c_1(p)(1 + |\gamma|)e^{\frac{\pi}{2}|\gamma|} \\ &\leq c_1(p)(1 + |\gamma|)e^{\frac{\pi}{2}|\frac{2}{p} - 1||\gamma|}e^{\frac{\pi}{2}(1 - |\frac{2}{p} - 1|)\gamma_p} \\ &\leq c_2(p)(1 + |\gamma|)e^{\frac{\pi}{2}|\frac{2}{p} - 1||\gamma|} \\ &\leq c_3(p)(1 + |\gamma|)^{|\frac{2}{p} - 1|}e^{\pi|\frac{1}{p} - \frac{1}{2}||\gamma|}, \end{aligned}$$

which proves the corollary in this case.

Otherwise, if $|\gamma| > \gamma_p$, then we define $\theta \in (0, 1)$ by

$$\theta = f(|\gamma|) \left| \frac{2}{p} - 1 \right|.$$

Further we can define q uniquely by

$$\frac{1}{p} = \frac{\theta}{q} + (1 - \theta) \frac{1}{2}.$$

Then we have

$$1 = f(|\gamma|) \left| \frac{2}{q} - 1 \right|,$$

and it follows from this that

$$\begin{aligned} |\gamma| \geq \frac{|\gamma|}{\log |\gamma|} &= \begin{cases} \frac{q}{2} & \text{if } 2 < q \\ \frac{q}{2q-1} & \text{if } 1 < q < 2 \end{cases} \\ &\geq \frac{1}{4} \left(q + \frac{1}{q-1} \right). \end{aligned}$$

By the Riesz-Thorin interpolation theorem we obtain from (6.6) and (6.7):

$$\begin{aligned} \| (-A)^{i\gamma} \|_{p,p} &\leq C_1^\theta \left[q + \frac{1}{q-1} \right]^\theta (1 + |\gamma|)^\theta e^{\frac{\pi}{2} |\gamma| \theta} \\ &\leq C_2 |\gamma|^\theta (1 + |\gamma|)^\theta e^{\frac{\pi}{2} |\frac{2}{p} - 1| |\gamma|} e^{\frac{\pi}{2} |\frac{2}{p} - 1| |\gamma| (f(|\gamma|) - 1)} \\ &= C_2 |\gamma|^\theta (1 + |\gamma|)^\theta e^{\frac{\pi}{2} |\frac{2}{p} - 1| |\gamma|} e^{\frac{\pi}{2} |\frac{2}{p} - 1| \log |\gamma| f(|\gamma|)} \\ &\leq C_2 (1 + |\gamma|)^{(1 + \frac{\pi}{2})\theta} e^{\frac{\pi}{2} |\frac{2}{p} - 1| |\gamma|} \\ &\leq C_3 (1 + |\gamma|)^{(4 + \pi) f(|\gamma|) |\frac{1}{p} - \frac{1}{2}|} e^{\frac{\pi}{2} |\frac{2}{p} - 1| |\gamma|}. \end{aligned}$$

Here we used that by the definition of f :

$$y (f(y) - 1) = y \frac{\log y}{y - \log y} = \log y f(y).$$

Since $f(|\gamma|) \leq \frac{e}{e-1}$, we have proved the assertion of the corollary. □

In his above cited paper [15] Cowling has a better estimate, but I could not figure out his computation. Furthermore he deduces, and we refer the reader to this deduction, the following theorem:

Theorem 6.3.2 (Cowling) *Let m be bounded, holomorphic in Γ_ψ , where $0 < \psi \leq \frac{\pi}{2}$. If p is such that $\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{\psi}{\pi}$, then for some constant C , not depending on p ,*

$$\| m(-A)f \|_p \leq C \left[\frac{\psi}{\pi} - \left| \frac{1}{p} - \frac{1}{2} \right| \right]^{-\frac{5}{2}} \| m \|_\infty.$$

Now we consider the family of functions

$$m_\theta : \lambda \mapsto \exp(e^\theta \lambda) \quad \theta \in \Gamma.$$

Then $m_0(tA) = e^{tA} = T_t$, and proving L^p -boundedness of the, on L^2 spectrally defined operators

$$m_\theta(tA) = e^{te^{i\theta}A} =: T_{te^{i\theta}}, \quad |\theta| < \eta,$$

will give an extension of the semigroup $(T_t)_{t \geq 0}$ to the complex sector Γ_η . Here, η is a possibly p -dependent constant, ideally we should find the largest possible one. The boundedness and the weak analyticity on $L^2(\Omega, \mu)$ imply the analyticity of this extension on this sector.

For this purpose the operators $(-A)^{i\gamma}$, $\gamma \in \mathbb{R}$ are quite useful. By inverting Mellin transforms we obtain for $x \leq 0$, $t \geq 0$:

$$\exp(te^{i\theta}x) - \exp(tx) = \frac{1}{2\pi} \int_{\mathbb{R}} [e^{-\gamma\theta} - 1] \Gamma(-i\gamma) t^{i\gamma} (-x)^{i\gamma} d\gamma. \quad (6.8)$$

Compute for this:

$$\begin{aligned} \int_0^\infty \exp(-e^{i\theta}\lambda) \lambda^{-i\gamma} \frac{d\lambda}{\lambda} &= \lim_{\varepsilon \searrow 0} \int_0^\infty \exp(-e^{i\theta}\lambda) \lambda^{\varepsilon-i\gamma-1} d\lambda \\ &= \lim_{\varepsilon \searrow 0} \int_0^\infty \exp(-\lambda) e^{-i\theta(\varepsilon-i\gamma)} \lambda^{\varepsilon-i\gamma-1} d\lambda \\ &= e^{-\gamma\theta} \int_0^\infty \lambda^{-i\gamma} e^{-\lambda} \frac{d\lambda}{\lambda} \\ &= e^{-\gamma\theta} \Gamma(-i\gamma). \end{aligned}$$

One can not directly make use of this last formula, since $|\Gamma(-i\gamma)| \sim \frac{1}{|\gamma|}$ for γ close to zero, and so for $x > 0$ an Integral

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-\gamma\theta} \Gamma(-i\gamma) t^{i\gamma} x^{i\gamma} d\gamma \simeq e^{-te^{i\theta}x}$$

does not converge. But in equation (6.8) it can be used that for some $C > 0$:

$$|e^{-\gamma\theta} - 1| |\Gamma(-i\gamma)| \leq e^{(|\theta| - \frac{\pi}{2})|\gamma|}. \quad (6.9)$$

This is used by Cowling to prove the following corollary due to Stein, c.f. [48] chap III, Theorem 1.

Corollary 6.3.2 (Stein) *If $0 \leq \theta \leq \frac{\pi}{2} - \pi \left| \frac{1}{p} - \frac{1}{2} \right|$, then the semigroup T_t extends to an analytic semigroup on Γ_θ .*

Sketch of a Proof: We may assume that $1 < p < \infty$, since otherwise the angle θ has to be zero.

The L^2 -spectral theory shows that the semigroup has a unique analytic extension on $L^2(\Omega, \mu)$, even to the cone $\Gamma_{\frac{\pi}{2}}$. We further obtain, for $z = te^{i\theta}$:

$$e^{te^{i\theta}A} - e^{tA} = \frac{1}{2\pi} \int_{\mathbb{R}} [e^{-\gamma\theta} - 1] \Gamma(-i\gamma) t^{i\gamma} (-A)^{i\gamma} d\gamma,$$

the integral being convergent in the strong operator topology, still on L^2 .

Now, if $f \in L^p(\Omega, \mu)$ and $g \in L^q(\Omega, \mu)$ ($\frac{1}{p} + \frac{1}{q} = 1$) are simple functions, then they are in L^2 . From the above inequality (6.9) and Corollary 6.3.1 we infer that for $|\psi| < \theta$:

$$\begin{aligned} | \langle T_{te^{i\psi}} f, g \rangle | &\leq \frac{C(p)}{2\pi} \int_{\mathbb{R}} [1 + |\gamma|^{12}]^{\frac{1}{p} - \frac{1}{2}} e^{\pi|\frac{1}{p} - \frac{1}{2}||\gamma|} e^{(|\psi| - \frac{\pi}{2})|\gamma|} d\gamma \|f\|_p \|g\|_q \\ &\leq \frac{C(p)}{2\pi} \int_{\mathbb{R}} [1 + |\gamma|^{12}]^{\frac{1}{p} - \frac{1}{2}} e^{-(\theta - |\psi|)|\gamma|} d\gamma \|f\|_p \|g\|_q \\ &\leq C_\psi \|f\|_p \|g\|_q. \end{aligned}$$

If general elements $f \in L^p(\Omega, \mu)$ and $g \in L^q(\Omega, \mu)$ are given, then they may be approximated by sequences of simple functions, $(f_k)_k$ and $(g_k)_k$ in the respective norms. The above estimate then shows that the sequence of analytic functions $z \mapsto \langle T_z f_k, g_k \rangle$ converges locally uniformly inside Γ_θ . \square

A formula similar to the above (6.8), together with the appropriate estimate:

$$e^{te^{i\theta}x} - \frac{1}{t} \int_0^t e^{sx} ds = \frac{1}{2\pi} \int_{\mathbb{R}} \left[e^{-\gamma\theta} - \frac{1}{1+i\gamma} \right] \Gamma(-i\gamma) t^{i\gamma} (-tx)^{i\gamma} d\gamma, \quad (6.10)$$

$$\left| e^{(-\theta\gamma)} - \frac{1}{1+i\gamma} \right| |\Gamma(-i\gamma)| \leq C e^{(|\theta| - \frac{\pi}{2})|\gamma|}, \quad (6.11)$$

can be used to prove a maximal theorem and an abstract non-tangential convergence theorem. (Cowling, in his paper proves an interesting extension of it):

Theorem 6.3.3 (Stein) *If $0 \leq \theta < \frac{\pi}{2}(1 - \left| \frac{2}{p} - 1 \right|)$, then for some constant $C_p > 0$ for all $f \in L^p(\Omega, \mu)$:*

$$\| \sup \{ |T_z f| : |z| < 1, z \in \Gamma_\theta \} \|_p \leq C_p \|f\|_p,$$

and

$$T_z f \rightarrow f \quad \mu\text{-almost everywhere as } z \rightarrow 0 \text{ in } \overline{\Gamma_\theta}.$$

Proof: We estimate pointwise

$$\begin{aligned} & \sup \{ |T_z f| : |z| < 1, z \in \Gamma_\theta \} \\ & \leq \sup_{|\psi| < \theta, 0 \leq t < 1} \left| \frac{1}{2\pi} \int_{\mathbb{R}} \left[e^{-\gamma\psi} - \frac{1}{1+i\gamma} \right] \Gamma(-i\gamma) t^{i\gamma} (-A)^{i\gamma} f d\gamma \right| + \\ & \quad \sup_{0 \leq t < 1} \left| \frac{1}{t} \int_0^t T_s f ds \right|. \end{aligned}$$

Thus the norm of the first term on the right hand side may be estimated using the above inequality (6.11) and Corollary 6.3.1:

$$\begin{aligned} & \left\| \sup_{|\psi| < \theta, 0 \leq t < 1} \frac{1}{2\pi} \int_{\mathbb{R}} \left| e^{-\gamma\psi} - \frac{1}{1+i\gamma} \right| |\Gamma(-i\gamma) t^{i\gamma}| |(-A)^{i\gamma} f| d\gamma \right\|_p \\ & \leq C \left\| \frac{1}{2\pi} \int_{\mathbb{R}} e^{(|\theta| - \frac{\pi}{2})|\gamma|} |(-A)^{i\gamma} f| d\gamma \right\|_p \\ & \leq C \frac{1}{2\pi} \int_{\mathbb{R}} e^{(|\theta| - \frac{\pi}{2})|\gamma|} \| (-A)^{i\gamma} f \|_p d\gamma \\ & \leq C \frac{1}{2\pi} \int_{\mathbb{R}} C(p) [1 + |\gamma|^3 \log^2 |\gamma|]^{\frac{1}{p} - \frac{1}{2}} e^{(|\theta| - \frac{\pi}{2})|\gamma| + \pi |\frac{1}{p} - \frac{1}{2}| |\gamma|} d\gamma \| f \|_p \\ & \leq C_p^1 \| f \|_p. \end{aligned}$$

Together with Theorem 5.4.3 this establishes the maximal inequality of the theorem.

The assertion on the almost everywhere convergence relies on an application of Banach's principle.

Theorem 6.3.4 (Banach's principle) *Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of operators on $L^p(\Omega, \mu)$, $1 \leq p < \infty$, such that for some constant $C > 0$, for all $f \in L^p(\Omega, \mu)$:*

$$\left\| \sup_{n \in \mathbb{N}} |T_n f| \right\|_p \leq C \| f \|_p.$$

If for all elements f in a dense subspace E of $L^p(\Omega, \mu)$

$$\lim_{n \rightarrow \infty} T_n f$$

exists pointwise almost everywhere, then for all $f \in L^p(\Omega, \mu)$

$$\lim_{n \rightarrow \infty} T_n f$$

exists pointwise almost everywhere.

Proof: We are given $f \in L^p(\Omega, \mu)$. For $\epsilon > 0$ we write $f = f_1 + f_2$ with $f_1 \in E$ and $\| f_2 \|_p < \epsilon$.

Then for μ almost all $\omega \in \Omega$

$$\begin{aligned}
0 &\leq \limsup_{n \rightarrow \infty} \operatorname{Re} T_n f(\omega) - \liminf_{n \rightarrow \infty} \operatorname{Re} T_n f(\omega) \\
&\leq \limsup_{n \rightarrow \infty} \operatorname{Re} T_n f_1(\omega) - \liminf_{n \rightarrow \infty} \operatorname{Re} T_n f_1(\omega) \\
&\quad + \limsup_{n \rightarrow \infty} \operatorname{Re} T_n f_2(\omega) - \liminf_{n \rightarrow \infty} \operatorname{Re} T_n f_2(\omega) \\
&\leq 0 + 2 \sup_{n \in \mathbb{N}} |T_n f_2|(\omega).
\end{aligned}$$

Hence

$$\left\| \limsup_{n \rightarrow \infty} \operatorname{Re} T_n f(\cdot) - \liminf_{n \rightarrow \infty} \operatorname{Re} T_n f(\cdot) \right\|_p \leq 2C\epsilon.$$

Since $\epsilon > 0$ is arbitrary this proves that

$$\limsup_{n \rightarrow \infty} \operatorname{Re} T_n f(\omega) - \liminf_{n \rightarrow \infty} \operatorname{Re} T_n f(\omega)$$

vanishes almost everywhere. An analogous argumentation for the imaginary parts then proves the almost everywhere convergence of the sequence $(T_n f)_{n \in \mathbb{N}}$. \square

The subspace E is exhibited using the analyticity of the semigroup in question. For this denote Γ_p the cone to which the semigroup extends analytically:

Lemma 6.3.1 *Given the conditions of Theorem 6.3.3. For $f \in L^p(\Omega, \mu)$ and $z \in \Gamma_p$ it is then possible to redefine $T_z f$ on a set of measure zero, such that for all $\omega \in \Omega$*

$$z \mapsto T_z f(\omega) \text{ is analytic on } \Gamma_p.$$

Proof: The weakly analytic map $\Phi : z \mapsto T_z f$, from Γ_p to $L^p(\Omega, \mu)$, is strongly analytic (See e.g. chap.5 sect.3 of [56]). Given a closed ball $B(z, r)$ of radius r with centre z contained in Γ_p , Φ has an, in the interior $B(z, r)^\circ$ of the ball, convergent expansion as a power series

$$T_\zeta f = \sum_{n=0}^{\infty} h_n (\zeta - z)^n \text{ with} \quad (6.12)$$

$$\sum_{n=0}^{\infty} \|h_n\|_p \rho^n < \infty \text{ for all } \rho < r. \quad (6.13)$$

For $n \in \mathbb{N}$ let $H_n : \Omega \rightarrow \mathbb{C}$ be a function representing h_n in $L^p(\Omega, \mu)$. From the estimate in (6.13) it follows that for $\rho < r$ $\sum_{n=0}^{\infty} |H_n(\omega)| \rho^n < \infty$ almost everywhere on Ω . In fact, the triangle inequality ensures that for all $k \in \mathbb{N}$

$$\left\| \sum_{n=0}^k |H_n| \rho^n \right\|_p \leq \sum_{n=0}^{\infty} \|H_n\|_p \rho^n < \infty.$$

The monotone convergence theorem then says that $\left(\sum_{n=0}^k |H_n| \rho^n\right)^p$ converges almost everywhere to an integrable function. Where this function is finite there $\sum_{n=0}^k |H_n| \rho^n$ must converge to a finite value too.

Choosing a sequence $\rho_k \nearrow r$ we find a set A_z of measure zero such that for $\omega \in \Omega \setminus A_z$ the series $\sum_{n=0}^{\infty} H_n(\omega)(\zeta - z)^n$ converges absolutely for all $\zeta \in B(z, r)^\circ$. On this ball let

$$F_z(\zeta, \omega) = \begin{cases} \sum_{n=0}^{\infty} H_n(\omega)(\zeta - z)^n & \omega \notin A_z \\ 0 & \omega \in A_z. \end{cases}$$

Now the (open) cone Γ_p can be covered by a countable union of balls $B(z_k, \frac{r_k}{2})^\circ$ with $B(z_k, r_k) \subset \Gamma_p$. If two such open balls have non-void intersection, say $B(z, \frac{r}{2})^\circ \cap B(z', \frac{r'}{2})^\circ \neq \emptyset$, then, for almost all $\omega \in \Omega$, the corresponding analytic functions $F_z(\zeta, \omega) = \sum_{n=0}^{\infty} H_n(\omega)(\zeta - z)^n$ and $F_{z'}(\zeta, \omega) = \sum_{n=0}^{\infty} H'_n(\omega)(\zeta - z')^n$ coincide on this intersection. To see this we may assume $r' \leq r$, then $z' \in B(z, r)^\circ$, and the series representing $F_z(\zeta, \cdot)$ converges outside A_z absolutely and uniformly in a neighbourhood of z' . Moreover, for $\omega \notin A_z \cup A_{z'}$:

$$H'_0(\omega) = \sum_{n=0}^{\infty} H_n(\omega)(z' - z)^n.$$

We can even differentiate (k times) to the result, that

$$\begin{aligned} H'_k(\omega) &= \frac{1}{k!} \left(\frac{d}{d\zeta}\right)^k \sum_{n=0}^{\infty} H'_n(\omega)(\zeta - z')^n|_{\zeta=z'} \\ &= \frac{1}{k!} \left(\frac{d}{d\zeta}\right)^k \sum_{n=0}^{\infty} H_n(\omega)(\zeta - z)^n|_{\zeta=z'} = \sum_{n=k}^{\infty} H_n(\omega) \binom{n}{k} (z' - z)^{n-k}. \end{aligned}$$

The last sum here is still absolutely convergent, and

$$\begin{aligned} \sum_{k=0}^{\infty} H'_k(\omega)(\zeta - z')^k &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} H_n(\omega) \binom{n}{k} (z' - z)^{n-k} (\zeta - z')^k \\ &= \sum_{n=0}^{\infty} H_n(\omega) \sum_{k=0}^n \binom{n}{k} (z' - z)^{n-k} (\zeta - z')^k \\ &= \sum_{n=0}^{\infty} H_n(\omega)(\zeta - z)^n. \end{aligned}$$

Now we define $T_\zeta f(\omega) = 0$ for $\omega \in \bigcup_k A_{z_k}$ and

$$T_\zeta f(\omega) = F_{z_k}(\zeta, \omega) \quad \text{for } \omega \in \Omega \setminus \bigcup_k A_{z_k},$$

where z_k is any element of our selection such that only $\zeta \in B(z_k, \frac{r_k}{2})^\circ$. □

We may now continue with the **proof of Theorem 6.3.3:**

From the strong continuity of the semigroup $(T_t)_{t \geq 0}$ at zero we see that $E = \{T_s f : f \in L^p(\Omega, \mu), s > 0\}$ is a dense linear subspace and for a sequence $z_n \in \Gamma_\theta$, $z_n \rightarrow 0$ we have for one of its elements $T_s f$ outside the set where $z \mapsto T_z f$ is not analytic:

$$\begin{aligned} \lim_{n \rightarrow \infty} T_{z_n} T_s f(\omega) &= \lim_{n \rightarrow \infty} T_{z_n + s} f(\omega) \\ &= T_s f(\omega). \end{aligned}$$

□

Remark 6.3.1

- (i) In the submarkovian case the angle of the cone of analyticity given here is not optimal. Liskevich and Perelmutter [35] obtain $\frac{\pi}{2} - \arctan \frac{|p-2|}{2\sqrt{p-1}}$.
- (ii) If one no longer assumes that the semigroup is contractive on the whole scale of L^p -spaces, then the above presented methods still apply.

If we assume that our strongly continuous semigroup fulfils, for some $1 < r < 2$ the following set of conditions:

- i') For those p such that $r \leq p \leq r'$ each operator T_t is a sub-positive contraction on $L^p(\Omega, \mu)$ ($\frac{1}{r} + \frac{1}{r'} = 1$),
- ii) Selfadjointness on L^2 : $T_t^* = T_t$ on $L^2(\Omega, \mu)$,
- iii) $T_0 = \text{id}$,

then, for $p < 2$, the angle of analyticity would be at least $\frac{\pi}{2} \frac{r}{p} \frac{2-p}{2-r}$. Of course we obtain the maximal theorem for the cone of this angle and the other consequences too.

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